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SOME MULTIPLIERS FOR THE $|C, 1|$ SUMMABILITY OF LAPLACE SERIES

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(Received : June 17, 1986 ; Revised : November 14, 1987)

ABSTRACT

In the present paper a new theorem on the absolute $(C, 1)$ summability of Laplace series has been proved.

1. INTRODUCTION

Let $f(\theta, \phi)$ be a function defined for the range $0 \leq \theta \leq \pi, 0 \leq \phi \leq 2\pi$ and integrable (L) over the surface of the sphere S . The Laplace series associated with this function is

$$f(\theta, \phi) \sim \frac{1}{2\pi} \sum_{n=0}^{\infty} \left(n + \frac{1}{2}\right) \iint_S f(\theta', \phi') P_n(\cos w) dS', \quad (1.1)$$

where

$$\begin{aligned} \cos w &= \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos(\phi - \phi') \\ dS' &= \sin \theta' d\theta' d\phi'. \end{aligned}$$

The Legendre polynomials $P_n(x)$ are defined by the relation

$$(1 - 2\mu x + \mu^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} P_n(x) \mu^n. \quad (1.2)$$

Following Kogbetliantz [3] the general value of $f(\theta, \phi)$ is

$$\phi(w) = \frac{1}{2\pi \sin w} \int_{C_w} f(\theta', \phi') ds', \quad (1.3)$$

where the curvilinear integral is taken along the small circle C_w , whose centre is (θ, ϕ) on the sphere and whose curvilinear radius is w . In view of the relation (1.3) the series (1.1) reduces to the form

$$\sum_{n=0}^{\infty} (n + \frac{1}{2}) \int_0^{\pi} \phi(w) P_n(\cos w) \sin w \, dw \equiv \sum_{n=0}^{\infty} A_n. \quad (1.4)$$

We write

$$\phi(t) = \int_t^{\delta} w^{-1,2} |\phi(w)| \, dw, \quad 0 < \delta \leq \pi$$

$$\beta_n = \prod_{\mu=1}^{k-1} (\log^{\mu} n) (\log^k n)^{1+\epsilon}, \quad \epsilon > 0$$

where

$$\log^k n = \log(\log^{k-1} n), \quad \dots, \log^2 n = \log \log n,$$

and prove the following

THEOREM. If $\{\lambda_n\}$ is a convex sequence such that $\sum_{n=1}^{\infty} \lambda_n/n < \infty$, and if

$$\phi(t) = O\left\{t^{\frac{1}{2}} (\log^k(1/t))^{\eta}\right\} \text{ as } t \rightarrow +0, \quad (1.5)$$

then the series

$$\sum \lambda_n A_n / \beta_n$$

is summable $|C, 1|$ for $0 < \eta < \epsilon$.

2. RESULTS REQUIRED

The following lemmas are required in our investigation :

Lemma 1.

$$|P_n(\cos \theta)| = O(1) \text{ for } 0 \leq \theta \leq \pi.$$

Lemma 2 [1].

$$\begin{aligned}
& -\frac{\pi}{2} \left[\frac{d}{dx} \{ P_{n+1}(x) + P_n(x) \}_{x=e^{i\theta} \cos \frac{1}{2} w} \right] \sin w \\
& = R \left\{ (n+1) e^{i(n+1)w + \pi/4} \left[\cot \frac{1}{2} w \right]^{\frac{1}{2}} \int_0^1 (1-u)^n u^{-\frac{1}{2}} g(w, u) du \right. \\
& \quad - (n+1) e^{i(n+3/2)w + \pi/4} (2 \sin w)^{-\frac{1}{2}} \int_0^1 (1-u)^n u^{-\frac{1}{2}} g(w, u) du \\
& \quad + e^{i(n+3/2)w + \pi/4} (2 \sin w)^{-\frac{1}{2}} \int_0^1 (1-u)^{n+1} u^{-\frac{1}{2}} g(w, u) du \\
& \quad \left. + \frac{1}{2} e^{i(n+5/2)w + 3\pi/4} (\sin \frac{1}{2} w)^{-1} (2 \sin w)^{-\frac{1}{2}} \int_0^1 (1-u)^{n+1} \right. \\
& \quad \quad \quad \left. u^{-\frac{1}{2}} [g(w, u)]^3 du \right. \\
& \quad \left. - e^{i(n+5/2)w + 3\pi/4} (2 \sin w)^{-3/2} \int_0^1 (1-u)^{n+1} u^{\frac{1}{2}} [g(w, u)]^3 du \right\},
\end{aligned}$$

where

$$g(w, u) = \{ 1 - ue^{i\theta} (2i \sin w)^{-1} \}^{-\frac{1}{2}} \leq \sqrt{2}.$$

Now let this be equal to $J_1 + J_2 + J_3 + J_4 + J_5$, say, where

$$J_1 = A \frac{(n+1) \Gamma(n+1)}{\Gamma(n+3/2)} (\cot \frac{1}{2} w)^{\frac{1}{2}} e^{i(n+1)w}$$

$$J_2 = A \frac{(n+1) \Gamma(n+1)}{\Gamma(n+5/2)} (\sin w)^{-\frac{1}{2}} e^{i(n+3/2)w}$$

$$J_3 = A \frac{\Gamma(n+2)}{\Gamma(n+5/2)} (\sin w)^{-\frac{1}{2}} e^{i(n+3/2)w}$$

$$J_4 = A \frac{\Gamma(n+2)}{\Gamma(n+5/2)} (\cos \frac{1}{2}w)^{-\frac{1}{2}} (\sin \frac{1}{2}w)^{-3/2} e^{i(n+5/2)w}$$

$$J_5 = A \frac{\Gamma(n+2)}{\Gamma(n+7/2)} (\sin w)^{-3/2} e^{i(n+5/2)w},$$

and A is a constant, not necessary the same at each occurrence.

3. PROOF OF THE THEOREM

Let τ_n be the n th Cesàro mean of first order of the sequence $\{n\lambda_n A_n/\beta_n\}$. It is sufficient to prove that

$$\sum_{n=2}^{\infty} |\tau_n|/n < \infty.$$

$$\tau_n = \frac{1}{n+1} \int_0^{\pi} \phi(w) \left\{ \sum_{v=2}^n \left(v + \frac{1}{2}\right) P_v(\cos w) \sin w \cdot v \lambda_v \beta_v^{-1} \right\} dw$$

$$= \int_0^{1/n} + \int_{1/n}^{\pi-(1/n)} + \int_{\pi-(1/n)}^{\pi}$$

$$= I_1 + I_2 + I_3, \text{ say.}$$

$$|I_1| = O(n^{-1}) \int_0^{1/n} w |\phi(w)| dw \cdot \sum_{v=2}^n v \left(v + \frac{1}{2}\right) \lambda_v \beta_v^{-1}$$

Now

$$\int_0^t |\phi(w)| dw = \int_0^t w^{\frac{1}{2}} \phi'(w) dw$$

$$\begin{aligned}
&= \left[w^{\frac{1}{2}} \phi(w) \right]_0^t - \frac{1}{2} \int_0^t w^{-\frac{1}{2}} \phi(w) dw \\
&= O \left[w (\log^k (1/w))^n \right]_0^t - O \left\{ \int_0^t \left[w^{\frac{1}{2}} (\log^k (1/w))^n \right] w^{-\alpha} dw \right\}, \\
&= O \{ t (\log^k (1/t))^n \}. \quad 0 < \alpha < 1
\end{aligned}$$

Hence

$$\begin{aligned}
|I_1| &= O(n \beta_n^{-1}) \int_0^{1/n} |\phi(w)| dw \sum_{v=2}^n \frac{\lambda_v}{v} \\
&= O \{ (\log^k n)^n / \beta_n \}. \quad (3.1)
\end{aligned}$$

By Abel's transformation,

$$I_2 = \frac{1}{n+1} \int_{1/n}^{\pi^{-1}(1/n)} \phi(w) \left\{ \sum_{v=0}^{n-1} \sum_{r=0}^v (r + \frac{1}{2}) P_r(\cos w) \sin w \Delta \right.$$

$$\left. \Delta (v \lambda_v \beta_v^{-1}) + \sum_{v=0}^n (v + \frac{1}{2}) P_v(\cos w) \sin w \cdot n \lambda_n \beta_n^{-1} \right\} dw$$

$$= \frac{1}{n+1} \int_{1/n}^{\pi^{-1}(1/n)} \phi(w) \left\{ \sum_{v=0}^{n-1} \frac{d}{dx} \left[P_{v+1}(x) + P_v(x) \right]_{x=\cos w} \right.$$

$$\left. \sin w \cdot \Delta (v \lambda_v \beta_v^{-1}) \right.$$

$$\left. + \frac{d}{dx} \left[P_{n+1}(x) + P_n(x) \right]_{x=\cos w} \sin w \cdot n \lambda_n \beta_n^{-1} \right\} dw$$

$$= I_{2,1} + I_{2,2}, \text{ say.}$$

$$I_{2,1} = \frac{1}{n+1} \int_{1/n}^{\pi^{-(1/n)}} \phi(w) \left\{ \sum_{v=0}^{n-1} [J_1 + J_2 + J_3 + J_4 + J_5] \Delta(v\lambda_v\beta_v^{-1}) \right\} dw$$

$$= I_{2,1,1} + I_{2,1,2} + I_{2,1,3} + I_{2,1,4} + I_{2,1,5}, \text{ say.}$$

As the other cases are similar, we consider only $I_{2,1,1}$ and $I_{2,1,4}$.

Since

$$\begin{aligned} \Delta(v\lambda_v\beta_v^{-1}) &= v\lambda_v \Delta\beta_v^{-1} + v\beta_{v+1}^{-1} \Delta\lambda_{v-\lambda_{v+1}} \beta_{v+1}^{-1} \\ &= O(\lambda_v\beta_v^{-1}) + O(v\beta_v^{-1} \Delta\lambda_v) \end{aligned}$$

therefore

$$\begin{aligned} |I_{2,1,1}| &= O(n^{-1}) \int_{1/n}^{\pi^{-(1/n)}} w^{-\frac{1}{2}} |\phi(w)| dw \left\{ \sum_{v=0}^{n-1} v^{\frac{1}{2}} \left[O(\lambda_v\beta_v^{-1}) + O(v\beta_v^{-1} \Delta\lambda_v) \right] \right\} \\ &= O\{n^{-3/2} (\log^k n)^n\} \left\{ O(n^{3/2} \beta_n^{-1}) \sum_{v=0}^{n-1} \frac{\lambda_v}{v} + O(n^{3/2} \beta_n^{-1}) \sum_{v=0}^{n-1} \Delta\lambda_v \right\} \\ &= O\{(\log^k n)^n / \beta_n\} \end{aligned}$$

and

$$\begin{aligned} |I_{2,1,4}| &= O(n^{-1}) \int_{1/n}^{\pi^{-(1/n)}} w^{-3/2} |\phi(w)| dw \left\{ \sum_{v=0}^{n-1} v^{-\frac{1}{2}} \left[O(\lambda_v\beta_v^{-1}) + O(v\beta_v^{-1} \Delta\lambda_v) \right] \right\} \\ &= O\left\{ n^{-\frac{1}{2}} (\log^k n)^n \right\} \cdot \left\{ O(n^{\frac{1}{2}} \beta_n^{-1}) \sum_{v=0}^{n-1} \frac{\lambda_v}{v} + O(n^{\frac{1}{2}} \beta_n^{-1}) \sum_{v=0}^{n-1} \Delta\lambda_v \right\} \\ &= O\{(\log^k n)^n / \beta_n\}. \end{aligned}$$

Hence

$$| I_{2,1} | = O \{ (\log^l n)^\tau / \beta_n \} .$$

$$\begin{aligned} I_{2,2} &= \frac{n \lambda_n \beta_n^{-1}}{n+1} \int_{1/n}^{\pi^{-(1/n)}} \phi(w) [J_1 + J_2 + J_3 + J_4 + J_5] dw \\ &= I_{2,2,1} + I_{2,2,2} + I_{2,2,3} + I_{2,2,4} + I_{2,2,5} , \text{ say .} \end{aligned}$$

$$| I_{2,2,1} | = O(n^{1/2} \lambda_n \beta_n^{-1}) \int_{1/n}^{\pi^{-(1/n)}} w^{-\frac{1}{2}} | \phi(w) | dw = O \left\{ \lambda_n (\log^k n)^\tau / \beta_n \right\} .$$

$$| I_{2,2,4} | = O(n^{-\frac{1}{2}} \lambda_n \beta_n^{-1}) \int_{1/n}^{\pi^{-(1/n)}} w^{-3/2} | \phi(w) | dw = O \left\{ \lambda_n (\log^k n)^\tau / \beta_n \right\} .$$

Therefore

$$| I_{2,2} | = O \{ \lambda_n (\log^k n)^\tau / \beta_n \} .$$

and hence

$$| I_2 | = O \{ (\log^l n)^\tau / \beta_n \} . \quad (3.2)$$

Lastly, we consider

$$\begin{aligned} | I_3 | &= O(n^{-1}) \int_{\pi^{-(1/n)}}^{\pi} | \phi(w) | \sin w dw \sum_{v=2}^n v(v+1) \lambda_v \beta_v^{-1} \\ &= O(n^{-1}) \int_0^{1/n} w dw \cdot O(n^3 \beta_n^{-1}) \sum_{v=2}^n \frac{\lambda_v}{v} \\ &= O(\beta_n^{-1}) . \end{aligned} \quad (3.3)$$

Combining (3.1), (3.2) and (3.3) , we obtain

$$| \tau_n | = O \{ (\log^k n)^\tau / \beta_n \} .$$

Therefore

$$\Sigma |\tau_n|/n = O \left\{ \sum \frac{1}{n \left[\prod_{\mu=1}^{k-1} \log^{\mu} n \right] (\log^k n)^{1+\epsilon-\eta}} \right\}$$

$$= O(1).$$

This completes the proof of the theorem .

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IPER - DUAL SYSTEMS AND HILBERT TRIPLETS

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ABSTRACT

We introduce the concept of iper-dual systems. Beginning from the iper-dual system $[H^*, S]$, we construct the Hilbert triplet $S \subseteq H \subseteq S'$, where S is a pre-Hilbert space dense in the (real or complex) Hilbert space H and S' is a space of linear functionals on S wider than H . The couple $[S', S]$ so obtained is an iperdual system. Once having introduced the order-bounded nets, we prove that if every weak-convergent net is order-bounded, then $S = H$. Finally we study four examples of special triplet $S \subseteq H \subseteq S'$.

0. INTRODUCTION

In Section 1 we introduce the concept of iper-dual systems, as a class of couples of linear spaces placed between the class of dual systems [1], and the class of Hilbert spaces.

We construct a space of linear functionals X' by means of the X -weak Cauchy sequences so that $X \subseteq \hat{X} \subseteq X'$, where $[\hat{X}, X]$ is a iper-dual system.

In Section 2 we consider the iper-dual system $[H^*, S]$, where S is a pre-Hilbert space, H is the Hilbert space completion of S and H^* is the dual of H . We construct the triplet $S \subseteq H \subseteq S'$ from $[H^*, S]$, in such a way that $[S', S]$ is an iper-dual system, too. The main result obtained in this section is that if every weak convergent net is order-bounded, then $S = H$.

Finally in Section 3 we study four examples of triplet $S \subseteq H \subseteq S'$. In the example 1, H is a separable, complex Hilbert space with a fixed orthonormal basis $\{u_n\}$, $S := Sp(\{u_n\})$ is the linear manifold spanned by $\{u_n\}$ and S' is constructed as described in Section 1. In this context we show that S' is identifiable with C^N , the space of all complex sequences.

In the example 2, $H = L_2(R^n)$ and S is the space $S(R^n)$ of functions of rapid decrease in R^n . In this case S' coincides with the space of tempered distributions.

In the example 3, $H = l_2$ and $S = \left\{ \{a_n\} : \sum_{n=1}^{\infty} |a_n|^2 (n+1)^m < \infty \forall m \in N \right\}$.

We show that $S' = \{ \{b_n\} : \exists c \in R \text{ and } \exists m \in N \text{ such that } |b_n| \leq c(1+n)^m \forall n \in N \}$, who coincides with the dual of S with respect to the topology of conutably normed space that makes S isomorphic to $S(R)$.

In the example 4, H is the space of Hilbert-Schmidt operators on a separable Hilbert space H and S is the space of finite rank operators on H . If we denote with $K(H)$ and $B(H)$ the space of compact operators and the space of bounded operators on H respectively, we show that $K(H) \subseteq S' \subseteq B(H)$.

It is useful to observe that in all previous examples, S is not complete in the topology induced by inner product. The fact that S is not complete is the necessary and sufficient condition in order that we obtain an *actual* enlargement S' of H^* .

1. IPER-DUAL SYSTEMS AND WEAK CAUCHY SEQUENCES

Let $\hat{X}, X$ be two linear spaces. We suppose that :

(IDS-1) X is a linear manifold of $\hat{X}$.

(IDS-2) There is a functional $[\cdot, \cdot] : \hat{X} \times X \rightarrow C$ such that

(a) the restriction of $[\cdot, \cdot]$ to $X \times X$ is an inner product on X denoted by $\langle \cdot, \cdot \rangle$.

(b) $[\hat{x}, \cdot]$ is a linear functional for every $\hat{x} \in \hat{X}$

(c) $[\cdot, x]$ is an antilinear functional for every $x \in X$.

(d) $[\hat{x}, x] = 0 \quad \forall x \in X \Rightarrow \hat{x} = 0$.

Set $\bar{R} = R \cup \{+\omega\}$. We define the function (called infi-norm)

$\|\cdot\|_\alpha : \hat{X} \rightarrow \bar{R}$ in the following manner :

$$\|\hat{x}\|_\alpha = \sup \{ |[\hat{x}, y]| : y \in X, \|y\| = 1 \}$$

where, of course, $\|y\|^2 = \langle y, y \rangle$. Then we have

$$(N-1) \quad \|\hat{x}\|_\alpha = 0 \quad \text{iff} \quad \hat{x} = 0$$

$$(N-2) \quad \|\lambda \hat{x}\|_\alpha = |\lambda| \|\hat{x}\|_\alpha \quad \forall \lambda \in C$$

$$(N-3) \quad \|\hat{x} + \hat{y}\|_\alpha \leq \|\hat{x}\|_\alpha + \|\hat{y}\|_\alpha$$

We remark that the generalized Schwartz inequality holds :

$$|[\hat{x}, x]| \leq \|\hat{x}\|_\alpha \|x\| \quad \forall \hat{x} \in \hat{X}, \quad \forall x \in X.$$

(of course, $\|x\|_\alpha = \|x\| \quad \forall x \in X$).

DEFINITION.

We say that $[\hat{X}, X]$ is an iper-dual system if (IDS-1),

(IDS-2) hold and moreover

(IDS-3) $\bar{X} \subseteq \hat{X}$, where $\bar{X}$ is the Hilbert space completion of

X with respect to the inner product $\langle \cdot, \cdot \rangle$.

Let $[X, X^\wedge]$ be an inner-product system. We say that a sequence

$\{\hat{x}_n\} \subseteq X^\wedge$ is X -weak Cauchy iff $\exists \lim_n [\hat{x}_n, y] \forall y \in X$.

We say $\{\hat{x}_n\}$ is X -weak convergent to $\hat{x} \in X^\wedge$ iff $\lim_n [\hat{x}_n, y] = [\hat{x}, y] \forall y \in X$.

We see immediately that the criterion of weak convergence defined on X is compatible with its linear structure [2].

Furthermore, we observe that there exist X -weak convergent sequences for which the corresponding sequences of inner-products do not converge to the inner-product of limit.

Let $L_w(X, X^\wedge)$ denote the linear space of the X -weak Cauchy sequences in $X^\wedge$. Every X -weak Cauchy sequence $\{\hat{x}_n\}$ defines a linear functional $t : X \rightarrow C$ by the law $\langle t, x \rangle := \lim_n [\hat{x}_n, x] \forall x \in X$.

The set of all linear functionals generated by some X -weak Cauchy sequence is denoted by X' . In the same way, generalizing the notion of X -weak Cauchy sequence, we shall say that a sequence $\{t_n\}$ in X' is a X -weak Cauchy sequence iff $\exists \lim_n \langle t_n, x \rangle \forall x \in X$.

Any X -weak convergent sequence in X' is a X -weak Cauchy sequence too. Now let $\{t_n\}$ be a X -weak Cauchy sequence in X' , then we can define the linear functional $l : X \rightarrow C$ as $\langle l, x \rangle = \lim_n \langle t_n, x \rangle \forall x \in X$ and we write $l = w - \lim t_n$.

In general l is not an element of X since it is not assured that there exists a sequence $\{\hat{x}_n\}$ in $X^\wedge$ which converges X -weak to l . In the next proposition we shall give a sufficient condition ensuring that $l \in X$:

PROPOSITION 1. Let $\{t_n\}$ be a X -weak Cauchy sequence in X' .

If any t_n is generated by a sequence $\{\hat{x}_{j(n)}^{(n)}\}_{j \in N}$ in $\hat{X}$ which is uniformly weak convergent to t_n in the sense that for every $n \in N$ and every $y \in X$ there exists an integer $j = j(n)$ depending on n but not on y , such that

$$| \langle t_n, y \rangle - [\hat{x}_{j(n)}^{(n)}, y] | \leq \frac{1}{n} \quad \forall y \in X$$

then $l = w\text{-}\lim t_n$ is an element of X' .

PROOF. Considering the sequence $\{\hat{x}_{j(n)}^{(n)}\}$ in $\hat{X}$ we get that it is a X -weak Cauchy sequence; indeed, from

$$| [\hat{x}_{j(n)}^{(n)}, y] - [\hat{x}_{j(m)}^{(m)}, y] | \leq | \langle t_n, y \rangle - [\hat{x}_{j(n)}^{(n)}, y] | +$$

$$| \langle t_n, y \rangle - \langle t_m, y \rangle | + | \langle t_m, y \rangle - [\hat{x}_{j(m)}^{(m)}, y] |$$

it follows that $\exists \lim [\hat{x}_{j(n)}^{(n)}, y] \quad \forall y \in X$.

On the other hand, the relation

$$| \langle l, y \rangle - [\hat{x}_{j(n)}^{(n)}, y] | \leq | \langle l, y \rangle - \langle t_n, y \rangle | + | \langle t_n, y \rangle - [\hat{x}_{j(n)}^{(n)}, y] |$$

under our hypotheses gives the result $l \in X'$.

On $L_w(\hat{X}, X)$ we can introduce the following equivalence relation, called the X -weak equiconvergence relation, defined as follows :

$$\{\hat{x}_n'\} \sim_w \{\hat{x}_n''\} \text{ iff } \lim_n [\hat{x}_n', y] = \lim_n [\hat{x}_n'', y] \quad \forall y \in X.$$

It is trivial to see that X' and $L_w(\hat{X}, X)/\sim_w$ are linearly isomorphic. Set $t \equiv [\{\hat{x}_n\}] \sim_w$. We suppose that $\|\hat{x}_n\|_\lambda < M$. If the sequence $\{y_k\} \subseteq X$ is norm-convergent to zero, then

$$0 \leq | \langle t, y_k \rangle | = \lim_n | \langle \hat{x}_n, y_k \rangle | \leq \sup_n \|\hat{x}_n\|_\lambda \cdot \|y_k\| \leq M \|y_k\|$$

so that $\lim_k \langle t, y_k \rangle = 0$. Thus t is sequentially continuous (and there-

fore continuous) with respect to the topology on X induced by the inner product. Hence there exists $x_0 \in \bar{X} \equiv \bar{X}^*$ such that $t = x_0$, where $\bar{X}^*$ denotes the dual space of $\bar{X}$ with respect to the topology induced by the inner product. Therefore we have proved that :

PROPOSITION 2.

If we denote by $L_w^b(\hat{X}, X)$ the linear subspace of $L_w(\hat{X}, X)$

of bounded sequences in inf-norm, then $\bar{X}$ and $L_w^b(\hat{X}, X)/\sim_w$

are linearly isomorphic. In particular, it follows that

$$\|x\|_\lambda < \infty \forall x \in \bar{X} \text{ and } \|\hat{x}\|_\lambda = +\infty \forall \hat{x} \in \hat{X} \setminus \bar{X}$$

2. HILBERT TRIPLETS AND WEAK CONVERGENT NETS

We shall construct now an iper-dual system by means of an other iper-dual system. In this Section, we denote by S a pre-Hilbert space with inner product $\langle \cdot | \cdot \rangle$. $(S, T_{\langle \cdot | \cdot \rangle}(S))$ denotes the locally convex topological vector space in which $T_{\langle \cdot | \cdot \rangle}(S)$ is the topology induced by the inner product. S^* denotes the dual space of $(S, T_{\langle \cdot | \cdot \rangle}(S))$ and H the Hilbert space completion of S . We denote by $D(S)$ the linear space antiisomorphic to S of all linear functionals $\langle x | : S \rightarrow C$, with $x \in S$, defined by $\langle x | : y \mapsto \langle x | y \rangle \forall y \in S$. We recall that $S = H$ iff $D(S) = S^*$. $[H^*, S]$ is an iper-dual system. We construct S' in the same manner as in the previous Section :

$$S' = \{t : S \rightarrow C : \exists \{x_n\} \in L_w(H^*, S), t = w\text{-}\lim_n \langle x_n \rangle\}.$$

We have that $S' \equiv L_w(H^*, S)/\sim_w$ and we have obtained the triplet

$$S \leq H \equiv H^* \equiv S' \subseteq S'$$

$[S', S']$ is an S' -dual system.

REMARK 1. It is easy to prove that the Hilbert space H^* is identifiable with $L_w^b(D(S), S)/\sim_w$ in which bounded S -weak Cauchy sequences from $D(S)$ are involved. We shall remember that a standard procedure for constructing the completion H^* of $D(S)$ is based on strong Cauchy sequences using the strong equiconvergence relation

$$\{x_n'\} \sim_s \{x_n''\} \text{ iff } \lim_n \|x_n' - x_n''\| = 0.$$

The above quoted result could seem very surprising, since S -weak Cauchy sequences are "much more" than strong Cauchy sequences from $D(S)$. However, it must be remarked that the S weak equiconvergence relation is weaker than the strong equiconvergence relation, so that S -weak equiconvergence classes are "larger" than strong equiconvergence classes.

REMARK 2. If $\{x_n\}$ is a sequence from H^* which is norm-bounded, then the existence of the $\lim_n \langle x_n | y \rangle \forall y \in S$ implies the existence of the

$\lim_n \langle x_n | t \rangle \forall t \in H$. However, the conditions

- (i) $\{x_n\} \sim_w \{x_n\}$
- (ii) $\exists \lim_n \langle x_n | t \rangle \forall t \in H$
- (iii) $\|x_n\| \leq M$

do not imply either the norm boundedness of $\{x_n\}$ or the existence of the $\lim_n \langle x_n | t \rangle \forall t \in H$.

We can endow the space S^* with the $\sigma(S^*, S)$ -topology generated by the basis of neighbourhoods of zero

$$U(x_1, \dots, x_n; \varepsilon) = \{l \in S^* : |\langle l, x_j \rangle| < \varepsilon \forall j = 1, \dots, n\},$$

$$x_1, \dots, x_n \in S, \varepsilon \in R_+$$

so in this way $(S^*, \sigma(S^*, S))$ is an Hausdorff locally convex space whose topological dual is S . We recall that $\sigma(S^*, S)$ is the weakest topology for S^* with respect to which all the element of S are continuous. Hence we have that $\sigma(S^*, S) \leq T_{<1>}(S^*)$.

The most important properties of the Hilbert triplet $S \subseteq H \equiv H^* \subseteq S'$ in relation to the corresponding iper-dual system are summarized in the following statements :

- (a) $(S, \sigma(S, S'))$ is a nuclear space ;
- (b) the canonical embedding $i : (S, \sigma(S, S')) \rightarrow (H, \sigma(H, H^*))$ is continuous;
- (c) H^* is dense in $(S', \sigma(S', S))$.

PROOF. (a) The topology $\sigma(S, S')$ for S is the projective topology with respect to the family $\{(C_\alpha, L_\alpha) : C_\alpha = C, L_\alpha S\}$. Since each $C_\alpha = C$ is nuclear, then the space $(S, \sigma(S, S'))$ is nuclear too [3] ,

(c) We shall make the identification $H \equiv H^*$. Let $t \in S'$. Then $t = w\text{-}\lim_n x_n$, $\{x_n\} \in L_w(H^*, S)$. We shall define $t_k := w\text{-}\lim_n \langle x_n^{(k)} |$

where $\{x_n^{(k)}\}$ is the sequence in $L_w^b(H^*, S)$ defined by $\{x_1, x_2, \dots, x_k, x_k, x_k, \dots\}$. It results that $\langle t_k, y \rangle = \langle x_k, y \rangle$ and so

$$\lim_k \langle t_k, y \rangle = \langle t, y \rangle \forall y \in S.$$

For sake of completeness we give now briefly some definitions and results useful in the sequel.

A partially ordered set I is said to be directed if, given $\alpha, \beta \in I$ there exists $\gamma \in I$ such that $\alpha \in \gamma$ and $\beta \in \gamma$.

A generalized sequence or a net in a topological space T is a mapping $f: I \rightarrow T$, where I is a directed set. As in the case of sequences, we write $f(\alpha) = x_\alpha$ and we represent the net by its image $\{x_\alpha\}$.

Given the net $\{x_\alpha\}$ in the topological vector space T , we say that $x \in T$ is the limit of $\{x_\alpha\}$ (and we write $x = \lim x_\alpha$) or that x_α converges to x if for any neighbourhood U_x of x there exists $\alpha_0 \in I$ such that $x_\alpha \in U_x \forall \alpha \geq \alpha_0$. It is well known that a function f from a topological space S to a topological space T is continuous iff for every convergent net $\{x_\alpha\}$ in S , with $x_\alpha \rightarrow x$, the net $f(x_\alpha)$ converges in T to $f(x)$. A net $\{x_\alpha\}$ in S is said to be order-bounded if there exists $\alpha_0 \in I$ such that $\{x_\alpha : \alpha \geq \alpha_0\}$ is norm-bounded.

The main result of this section is the following

THEOREM. (a) $S = H \Rightarrow L_w(H^*, S) = L_w^c(H^*, S)$

(b) If every net $\sigma(S, D(S))$ -convergent is order-bounded, then $S = H$.

PROOF. (a) It is an obvious consequence of the Uniform Boundedness Principle [4].

(b) It is sufficient to prove that $D(S) = S^*$. Since the topological dual of $(S, \sigma(S, D(S)))$ is $D(S)$, it is sufficient to show that every $l \in S^*$ is $\sigma(S, D(S))$ -continuous on S . For this, let $l \in S^*$ and let $\{x_\alpha\}$ be a net $\sigma(S, D(S))$ -convergent to x . Since $l \in S^*$, there exists $\{y_k\} \subseteq D(S)$ such that $\|L \cdot y_k\| \rightarrow 0$ as $k \rightarrow \infty$. Therefore

$$|\langle l, x_\alpha - x \rangle| = |\langle l \pm y_k, x_\alpha - x \rangle| \leq \|l - y_k\| (\|x_\alpha\| + \|x\|) + |\langle y_k, x_\alpha - x \rangle|.$$

Since $\{x_\alpha\}$ is order-bounded, there exist an $\alpha_0 \in I$ and a real number $M > 0$ such that $\|x_\alpha\| < M$ for every $\alpha \geq \alpha_0$. So, $\varepsilon > 0$, there exists $k_0 \in N$ such that $\|y_{k_0} - l\| < \varepsilon/2 (M + \|x\|)$. Moreover, as $\{x_\alpha\}$ is $\sigma(S, D(S))$

- convergent to x , there exists $\bar{\alpha} \in I$ such that $|\langle y_{k0} | x_{\alpha} - x \rangle| < \epsilon/2$ for every $\alpha \geq \bar{\alpha}$.

Let $\alpha_1 \geq \{\alpha_0, \bar{\alpha}\}$. Then for every $\alpha \geq \alpha_1$ we have

$$|\langle l, x_{\alpha} - x \rangle| \leq \|l - y_{k0}\|(\|x_{\alpha}\| + \|x\|) + |\langle y_{k0} | x_{\alpha} - x \rangle| < \frac{\epsilon}{2(M + \|x\|)}(H + \|x\|) + \frac{\epsilon}{2} = \epsilon, \text{ i.e. } \lim \langle l, x_{\alpha} \rangle = \langle l, x \rangle.$$

REMARK. The statement (b) has a converse when H is finite-dimensional (i.e. in a finite-dimensional Hilbert space every weakly convergent net is order-bounded), but the author ignores whether this property characterizes any Hilbert space.

3. EXAMPLES

EXAMPLE 1. We denote by H a separable, infinite-dimensional complex Hilbert space, by $\{u_n\}$ an orthonormal basis fixed in H and by $S = Sp(\{u_n\})$ the linear manifold spanned by $\{u_n\}$.

Let $E_k : H \rightarrow H$ be the projection on the finite-dimensional subspace $S^k := Sp(u_1, \dots, u_k)$:

$$E_k x := \sum_{j=1}^k \langle u_j | x \rangle u_j \quad \forall x \in H.$$

It is straightforward to verify that the functional $\langle 1 \rangle_k : S \times S \rightarrow \mathbb{C}$ defined by

$$\langle x | y \rangle_k := \langle E_k x | E_k y \rangle = \sum_{j=1}^k \langle x | u_j \rangle \langle u_j | y \rangle$$

is a degenerate inner-product on S . We shall denote S_k the space S endowed with the degenerate inner product $\langle 1 \rangle_k$.

The function $p_k : x \rightarrow p_k(x) = \left(\sum_{j=1}^k |\langle u_j | x \rangle|^2 \right)^{1/2}$ is a seminorm on S .

Of course for the family of seminorms $F(\{u_n\}) = \{p_k : k \in \mathbb{N}\}$ holds $p_k(x) \leq p_{k+1}(x) \forall k \in \mathbb{N}$ and $\forall x \in S$. In the previous Section we have endowed the pre-Hilbert space S with the weak topology $\sigma(S, D(S))$, that is of the topology generated by the family of seminorms $F(D(S)) = \{q_k : k \in S\}$.

We show now that in the present example we have that

$F(D(S))$ is equivalent to the totally ordered family of seminorms $F(\{u_n\})$.

PROOF. For every $q_x, x \in S$, we can write $x = \sum_{j=1}^n \alpha_j u_j$ and so for every $y \in S$,

$$q_x(y) = \left| \left\langle \sum_{j=1}^n \alpha_j u_j \mid y \right\rangle \right| \leq \sum_{j=1}^n |\alpha_j| |\langle u_j \mid y \rangle|.$$

If $A = \max(|\alpha_j|)$, we have that $q_x(y) \leq A \sum_{j=1}^n |\langle u_j \mid y \rangle| \leq A \sum_{j=1}^n p_j(y)$.

Moreover, $(p_k(x))^2 = \sum_{j=1}^k |\langle u_j \mid x \rangle|^2 \leq \left(\sum_{j=1}^k |\langle u_j \mid x \rangle| \right)^2$

$$= \left(\sum_{j=1}^k q_{u_j}(x) \right)^2 \Rightarrow p_k(x) \leq \sum_{j=1}^k q_{u_j}(x).$$

Finally, we construct S' . We recall that

$$S' = \{ t : S \rightarrow \mathbb{C} : \exists \{x_n\} \in L_w(H^*, S) \text{ such that}$$

$$\langle t, y \rangle = \lim_n \langle x_n \mid y \rangle \quad \forall y \in S \}.$$

Since $S = Sp(\{u_n\})$, we have that

Let $t \in S'$. If $\alpha_j = \langle t | u_j \rangle \quad \forall j \in N$ and $y \in S$, $y = \sum_{i=1}^r \lambda_i u_i$, then

$$\langle t, y \rangle = \sum_{i=1}^r \lambda_i \alpha_i.$$

The (*)¹ suggest to put, formally, $t = \sum_{j=1}^{\infty} \alpha_j u_j$, so we can identify t with the sequence $\{\alpha_j\}$.

The map $\phi : S' \rightarrow C^N$, defined for every $t = \sum_{j=1}^{\infty} \alpha_j u_j$ in S' by $\phi(t) = \{\alpha_j\}$ is a linear isomorphism between S' and C^N .

In fact it is easy to verify that ϕ is linear and injective. If $\{\alpha_j\} \in C^N$

we consider the sequence $\{x_n\} \subseteq S$ defined by $x_n = \sum_{j=1}^n \alpha_j u_j$. It is easy

to see that $\{x_n\} \in L_w(H^*, S)$ and if $t = w\text{-}\lim_n x_n$, then $\phi(t) = \{\alpha_j\}$, thus ϕ is surjective.

EXAMPLE 2. If H is the Hilbert space $L_2(R^n)$ of square Lebesgue-integrable complex-valued functions on R^n and S is the space $T(R^n)$ dense in $L_2(R^n)$ of the functions of rapid decrease on R^n , then S is the space of tempered distributions $T'(R^n)$ according to the definition in [5]. We have so obtained the iper-dual system $[T'(R^n), T(R^n)]$ and the Hilbert triplet $T(R^n) \subseteq L_2(R^n) \subseteq T'(R^n)$.

EXAMPLE 3. Let $H = l_2$ and $S = T$, where

$$T = \{ \{a_n\} \in l_2 : \sum |a_n|^2 (n+1)^m < \infty \quad \forall m \in N \}.$$

We show that S' coincides with the pual of T endowed with a suitable topology. If $m \in N$ is fixed, we define the norm $\| \cdot \|_m$ with

$$\| \underline{a} \|_m := \left(\sum |a_n|^2 (n+1)^{1/2} \right)^{1/2} \quad \forall \underline{a} = \{a_n\} \in T.$$

The family of norms $(\|\cdot\|_m)_{m \in N}$ is a directed family of norms, that is $\forall k_1, \dots, k_r \in N, \exists k \in N$ and $\exists c \in R_+$ such that $\|a\|_{k_1} + \dots + \|a\|_{k_r} \leq c \|a\|_k \forall a \in T$ since, put $k = \max(k_1, \dots, k_r)$, we have $\|a\|_{k_1} + \dots + \|a\|_{k_r} \leq r \|a\|_k$.

We consider now the countably normed space $(T, \|\cdot\|_m)$. It is well known that a sequence $(\underline{a}^{(n)})_{n \in N}$ in $(T, \|\cdot\|_m)$ converges to zero iff $\lim_n \|\underline{a}^{(n)}\|_m = 0 \forall m \in N$. It is also easy to prove that $(T, \|\cdot\|_m)$ is a complete countably normed space.

The proof of following lemmas 1 and 2 are in [6]:

LEMMA 1. A linear functional $t : (T, \|\cdot\|_m) \rightarrow C$ is continuous iff $\exists c \in R_+$ and $\exists k \in N$ such that

$$|t(\underline{a})| \leq c \|\underline{a}\|_k \quad \forall \underline{a} \in T.$$

LEMMA 2. (UNIFORM BOUNDEDNESS PRINCIPLE IN COUNTABLY NORMED SPACES). Let X be a complete countably normed space with $\|\cdot\|_r$ a directed sequence of norms. Let F be a set in X^* , the dual of X . If $\{F(f) : F \in F\}$ is bounded for each $f \in X$ then there is a $c \in R_+$ and an $r \in N$ so that, for all $f \in X$ and all $F \in F$,

$$|F(f)| \leq c \|f\|_r.$$

Utilizing the Lemmas 1 and 2 it is now easy to prove the

LEMMA 3. Let $(\underline{x}^{(n)}) \subseteq l_2$ be a sequence in l_2 such that $\exists \lim_n \langle \underline{a}, \underline{x}^{(n)} \rangle$ $\forall \underline{a} \in T$. Then, if $t = w\text{-}\lim_n \langle \underline{x}^{(n)} \rangle$, we have that $t \in (T, \|\cdot\|_m)'$, where $(T, \|\cdot\|_m)'$ is the topological dual of the countably normed space $(T, \|\cdot\|_m)$.

LEMMA 4. Let $(\underline{e}^{(n)})$ be the canonical basis in l_2 ($\underline{e}^{(n)} = \{\delta_{kn}\}_{k \in N}$).

Let $T \in (T, \|\cdot\|_m)'$, $b_n = T(e^{(n)})$. Then there is a $c \in \mathbb{R}$ and a $k \in \mathbb{N}$ so that $|b_n| \leq c(1+n)^m$ and

$$(*)^3 \quad T a = \sum_{n=1}^{\infty} a_n b_n \quad \forall a \in T,$$

Conversely, if $\|b_n\| \leq c(1+n)^m$, then $(*)^3$ defines an element of $(T, \|\cdot\|_m)'$.

PROOF. If $T \in (T, \|\cdot\|_m)'$, then from lemma 1 follows $|b_n| \leq c(1+n)^m$ and moreover the $(*)^3$ is trivial. Conversely, if $|b_n| \leq c(1+n)^m$ then

$$\begin{aligned} |\sum a_n b_n|^2 &\leq (\sum |a_n|^2 (n+1)^{2m+2}) (\sum |b_n|^2 (n+1)^{-(2m+2)}) \\ &\leq 1/6 c^2 \pi^2 \|a\|_{2m+2}^2 \end{aligned}$$

and so, still from lemma 1, the $(*)^3$ defines an element of $(T, \|\cdot\|_m)'$.

THEOREM. In the previous notations, we have $S' = (T, \|\cdot\|_m)'$,

PROOF. $S' \subseteq (T, \|\cdot\|_m)'$ is the Lemma 3.

Conversely, if $T \in (T, \|\cdot\|_m)'$, then from Lemma 4, if $b_n = T(e^{(n)})$, it results $|b_n| \leq c(1+n)^m$ and $T(a) = \sum a_n b_n$. But then, put $x^{(n)} = (b_1, \dots, b_n, 0, 0, \dots)$ we have that $\forall a \in T, \exists \lim_n \langle x^{(n)} | a \rangle = \sum b_n a_n = T(a)$

We have thus obtained the iper-dual system $[(T, \|\cdot\|_m)', T]$ and the Hilbert triplet $\{\{a_n\} : \sum |a_n|^2 (n+1)^n < \infty \forall m\}$

$\subseteq \{\{a_n\} : \sum |a_n|^2 < \infty\} \subseteq \{\{a_n\} : \exists c \in \mathbb{R}_+, \exists m \in \mathbb{N}, |a_n| \leq c(1+n)^m\}$.

REMARK. If $(T(R), \|\cdot\|_{b,h})$ denotes the countably normed space of functions of rapid decrease on R with $\|\cdot\|_{b,h}$ defined by $\|\Phi\|_{b,h} := \|x^k \Phi^{(h)}(x)\|_{\infty}$, where $\|\Phi\|_{\infty} = \sup_{x \in R} |\Phi(x)|$, it is well known that

the countably normed space $(T(R), \|\cdot\|_{k,h})$ and $(T, \|\cdot\|_m)$ are isomorphic ([6]). This provides, "en passant", a proof of equivalence of definitions of tempered distributions via linear functionals $([f])$ and via weakly Cauchy sequences [5].

EXAMPLE 4. Let H be a separable Hilbert space, Let $B(H)$ be the space of bounded operators on H . We are interested to the following linear varieties of $B(H)$:

$$\begin{aligned} F(H) &:= \{ \text{finite rank operators} \} \\ HS(H) &:= \{ \text{Hilbert-Schmidt operators} \} \\ K(H) &:= \{ \text{compact operators} \} \end{aligned}$$

We recall that [8]:

- (1) $F \in F(H)$ iff the range $R(F)$ of F is finite-dimensional.
- (2) $K \in K(H)$ iff $\exists \{v_i\}, \{w_i\}$ orthonormal bases of H and $\exists \{\mu_i\} \subseteq R_+$ with $\mu_1 \geq \mu_i \forall i \in N$ and $\lim_i \mu_i = 0$, so that

$$(*)_1^4 \quad Kx = \sum_{i=1}^{\infty} \mu_i \langle x | v_i \rangle w_i \quad \forall x \in H.$$

- (3) $T \in HS(H)$ iff $T \in K(H)$ and the sequence $\{\mu_i\}$ of $(*)_1^4$ is in l_2 ,
- (4) $T(H) \subset HS(H) \subseteq K(H) \subseteq B(H)$.
- (5) If $T_1, T_2 \in HS(H)$ and $\{u_i\}$ is an orthonormal basis of H , the numerical series $\sum_{i=1}^{\infty} \langle T_1 u_i | T_2 u_i \rangle$ is absolutely convergent and the sum is independent from the orthonormal basis $\{u_i\}$.
- (6) $HS(H)$ can be endowed with the inner product $\langle T_1 | T_2 \rangle := \sum \langle T_1 u_i | T_2 u_i \rangle$ where $\{u_i\}$ is any orthonormal basis in H . With respect to this inner product, $HS(H)$ is a separable Hilbert space, which we denote by H .

- (7) If we denote with S the space $F(H)$, it results that S is dense in H . So we can construct S' as in Section 2.

On the contrary of example 1 in which beginning from the space S of finite sequences we have obtained that S' is the space of *all* sequences, now, beginning from the space S of finite rank operators, we obtain that $S' \subseteq B(H)$ and so it is not made up of *all* linear operators on H , as it results from the following theorem :

THEOREM In the previous notations, we have $K(H) \subseteq S' \subseteq B(H)$.

PROOF. Let $K \in K(H)$. Then $Kx = \sum_1^\infty \mu_i \langle x | v_i \rangle w_i$, $\forall x \in H$. We

can look to K as a functional on S defined by

$$K(F) := \sum_{i=1}^s \lambda_i \langle K v_i | \tilde{w}_i \rangle \quad \forall F \in S, \quad Fx = \sum_{i=1}^s \lambda_i \langle x | \tilde{v}_i \rangle \tilde{w}_i.$$

For each $n \in N$ we define $K_n \in H$ with

$$K_n x = \sum_{i=1}^n \mu_i \langle x | v_i \rangle w_i,$$

and we show that $K = w \lim \langle K_n |$.

Indeed if $F \in S$, $Fx = \sum_{i=1}^s \lambda_i \langle x | \tilde{v}_i \rangle \tilde{w}_i$,

$$(*)_2^4 \quad \langle K_n | F \rangle = \sum_{i=1}^\infty \langle K_n \tilde{v}_i | F \tilde{v}_i \rangle = \sum_{i=1}^s \lambda_i \langle K_n \tilde{v}_i | \tilde{w}_i \rangle$$

and so $\lim_n \langle K_n | F \rangle = K(F)$. In this way we have proved that $K(H) \subseteq S'$.

We show now that $S' \subseteq B(H)$. Let $t = w\text{-}\lim_n \langle T_n \in HS(H) \rangle$. From $(*)_2^4$

it is easy to see that

$$(*)_3^4 \quad \exists \lim_n \langle T_n x | y \rangle \quad \forall x, y \in \mathbf{H}$$

(in effect this is a necessary and sufficient condition in order that $\exists \lim_n \langle T_n | F \rangle \quad \forall F \in S$).

Of course the $(*)_3^4$ is equivalent to

$$(*)_4^4 \quad \exists \lim_n \langle T_n^* x | y \rangle \quad \forall x, y \in \mathbf{H}$$

(we recall that $T \in HS(\mathbf{H}) \Rightarrow T^* \in HS(\mathbf{H})$. where T^* denotes the adjoint of T).

Now from $(*)_3^4$, once fixed x , from the Uniform Boundedness Principle and from the Fréchet-Riesz theorem, it follows that $\exists ! Tx \in \mathbf{H}$ such that

$$\lim_n \langle T_n x | y \rangle = \langle Tx | y \rangle \quad \forall y \in \mathbf{H} .$$

As x run over $\mathbf{H}$, we obtain a map $T : \mathbf{H} \rightarrow \mathbf{H}$ which is linear, as it is easy to verify .

Analogously from $(*)_4^4$ we obtain a linear mapping T satisfying

$$(*)_5^4 \quad \lim_n \langle \tilde{T}_n^* x | y \rangle = \langle \tilde{T} x | y \rangle \quad \forall x, y \in \mathbf{H}$$

There remain to prove that $T \in \mathbf{B}(\mathbf{H})$. We recall that is sufficient to prove that T has an adjoint operator defined on $\mathbf{H}$, [4] .

This is immediately verified once noted that the adjoint of T is the

$$\begin{aligned} \text{operator } \tilde{T} \text{ since } \langle Tx | y \rangle &= \lim_n \langle T_n x | y \rangle = \lim_n \langle x | T_n^* y \rangle \\ &= \langle x | \tilde{T} y \rangle \quad \forall x, y \in \mathbf{H} . \end{aligned}$$

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ON SOME FIXED POINT THEOREMS

By

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There has been given a series of extensions of the well-known Banach Contraction Principle. Caristi [1] gave an important theorem where the function is not of contraction type, not even continuous. In this paper we prove a theorem where the function is not necessarily continuous and then we derive several known results as corollaries.

We need the following terminologies .

Let X be a metric space and $f : X \rightarrow X$ be a self mapping . Then, for $x \in X$, $O(x, \infty) = (x, fx, f^2x, \dots)$ is called the orbit of x . X is said to be orbitally complete if each Cauchy sequence in $O(x, \infty)$ converges there. If X is a complete metric space then it is orbitally complete, however, an orbitally complete metric space is not necessarily complete .

Let us recall the following well-known results.

Theorem 1. Let $f : X \rightarrow X$ be a contraction map on a complete metric space. Then f has a unique fixed point .

Theorem 2. Let $f : X \rightarrow X$ and $\Phi : X \rightarrow [0, \infty)$, where X is a complete metric space and Φ is a lower semicontinuous function. If for each $x \in X$

$$d(x, fx) \leq \Phi x - \Phi(fx)$$

then f has a fixed point.

Note that f is not even continuous in this case.

We state our main result as follows .

Theorem 3. Let $f: X \rightarrow X$ be a function, where X is an orbitally complete metric space , Assume further that

- (i) for each $\epsilon > 0$ there is a $\delta(\epsilon) > 0$ such that if $d(x, fx) < \delta(\epsilon)$ then $f[B(x, \epsilon)] \subset B(x, \epsilon)$, and
- (ii) $d(f^n x_0, f^{n+1} x_0) \rightarrow 0$ as $n \rightarrow \infty$ for some $x_0 \in X$.
Then the sequence $\{f^n x_0\}$ converges to a fixed point of f ,

Proof . Let $O(x, \infty) = \{x, fx, f^2x, \dots\}$. We want to show that $\{f^n x_0\}$ is a Cauchy sequence .

Given $\epsilon > 0$ we choose N such that $d(f^n x_0, f^{n+1} x_0) < \delta(\epsilon)$ for all $n \geq N$. For simplicity, we set $x_n = f^n x_0$, for all $n \in \mathbb{N}$.

Now $d(x_N, x_{N+1}) = d(x_N, fx_N) < \delta(\epsilon)$ so we get $f[B(x_N, \epsilon)] \subset B(x_N, \epsilon)$. In particular $x_{N+1} \in B(x_N, \epsilon)$. By induction, we get that $f^i x_N = x_{N+i} \in B(x_N, \epsilon)$ for all $i \geq 0$. Thus $d(x_i, x_r) \leq 2\epsilon$ for $i, r \geq N$. This shows that $\{f^n x_0\}$ is a Cauchy sequence . Since X is an orbitally complete metric space $\{f^n x_0\}$ converges to some x in X .

We show that $fx = x$.

If $x \neq fx$, then $d(x, fx) = a > 0$. We seek a contradiction .

We choose n sufficiently large such that $x_n = f^n x_0 \in B(x, a/3)$ and $d(x_n, x_{n+1}) < \delta(a/3)$. So $f[B(x_n, a/3)] \subset B(x_n, a/3)$. This gives that $fx \in B(x_n, a/3)$.

However,

$$d(x_n, fx) \geq d(fx, x) - d(x, x_n) \geq (2/3) a .$$

So $fx \notin B(x_n, a/3)$, a contradiction .

This implies that $d(x, fx) = 0$, i. e., x is a fixed point of f ,

Now, we derive several known results as corollaries .

Remark. If $f: X \rightarrow X$ is a contraction map, i. e., $d(fx, fy) \leq kd(x, y)$ for any $x, y \in X$, where $0 \leq k < 1$, then (i) and (ii) of Theorem 3 are satisfied. Indeed, for any given $\varepsilon > 0$, let $\delta = (1-k)\varepsilon$. Now, if $d(x, fx) < \delta$, then for $z \in B(x, \varepsilon)$ we have that

$$\begin{aligned} d(fz, x) &\leq d(fz, fx) + d(fx, x) \leq kd(z, x) + d(fx, x) \\ &< k\varepsilon + (1-k)\varepsilon = \varepsilon \end{aligned}$$

Hence (i) is satisfied.

Moreover,

$$d(f^n x, f^{n+1} x) \leq k^n d(x, fx) \quad \text{for any } x \in X.$$

Hence (ii) is satisfied, too.

We close this note with the following result.

Corollary (see [2]). Let $f: X \rightarrow X$ satisfy the condition

$$d(fx, fy) \leq \Phi(d(x, y))$$

where X is an orbitally complete metric space, $\Phi: R^+ \rightarrow R^+$ is a monotone, increasing function such that $\Phi^n(t) \rightarrow 0$ for each fixed $t > 0$. Then f has a unique fixed point x , and $x_n = f^n x$ converges to x .

Proof. Clearly $d(f^n x, f^{n+1} x) \rightarrow 0$ as $n \rightarrow \infty$. Now, since $d(f^n x, f^{n+1} x) \leq \Phi^n(d(x, fx))$, let $\varepsilon > 0$ be given. Choose $\delta = \varepsilon - \Phi(\varepsilon)$. If $d(x, fx) < \delta$ then for $z \in B(x, \varepsilon)$ we get $d(x, fz) \leq d(x, fx) + d(fx, fz) < \varepsilon - \Phi(\varepsilon) + \Phi(d(x, z))$. Being Φ monotone increasing, we have that $f(z) \in B(x, \varepsilon)$, so $f[B(x, \varepsilon)] \subset B(x, \varepsilon)$. Then the result follows from Theorem 3.

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FIXED AND COMMON FIXED POINTS IN CONVEX METRIC SPACES

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Recently many authors have proved fixed point theorems) see for example [1], [3], [5], [9]) for operators mapping a Banach space X into itself. In each of these theorems the operators involved are nonexpansive *i. e.* if T maps the Banach space X into itself, then

$$(a) \quad \|Tx - Ty\| \leq \|x - y\|, \text{ for } x, y \in X$$

Kannan [7], [8] proved some fixed point theorems for operators mapping a Banach space into itself which, instead of being nonexpansive possess the following : if T is a mapping of a Banach space X into itself, then

$$(b) \quad \|Tx - Ty\| \leq 1/2 [\|x - Tx\| + \|y - Ty\|] \text{ for } x, y \in X.$$

It may be observed that condition (a) implies the continuity of the operators in the whole space while condition (b) has no such implication.

The main aim of the present paper is to generalize the results of Kannan [7], [8] to convex metric spaces. We also extend the results of Kirk [9], Goebel, Kirk and Shimi [6] to the setting of convex metric spaces.

In the sequel we use I for the interval $[0, 1]$. The following notion was introduced by Takahashi [11].

Definition 1.1. Let X be a metric space. A mapping $W : X \times X \times I \rightarrow X$ is said to be a *convex structure* on X if for all $x, y \in X$ and $\lambda \in I$ the following condition is satisfied :

$$d(u, W(x, y; \lambda)) \leq \lambda d(u, x) + (1-\lambda) d(u, y)$$

for all u in X . We will call a metric space with a convex structure a *convex metric space*.

If X is a Banach space, then, as a metric space with $d(x, y) = \|x - y\|$, the mapping $W : X \times X \times I \rightarrow X$ defined by $W(x, y; \lambda) = \lambda x + (1-\lambda)y$ is a convex structure,

Definition 1.2. Let X be a convex metric space. Let K be a nonempty subset of X . we say that K is *convex* if $W(x, y; \lambda)$ belongs to K for all x, y in K and $\lambda \in I$.

Definition 1.3. Let X be a metric space and E be a nonempty subset of X , then we define

$$R_x(E) = \sup \{ d(x, y) : y \in E \}$$

$$R(E) = \inf \{ R_x(E) : x \in E \}$$

$$E_C = \{ x \in E : R_x(E) = R(E) \}.$$

Definition 1.4. Let X be a convex metric space. We say that X has *property (C)* if every bounded decreasing net of nonempty closed and convex subsets of X has a nonempty intersection.

It follows from Smulian's theorem that every weakly compact convex subset of a Banach space has property (C) [4, p. 433].

Lemma 2.1. Let X be a convex metric space. If X has property (C), then E_C is nonempty, closed and convex.

Proof. For each $n \in N$, let $C(x, n) = \{y \in E : d(x, y) \leq R(E) + \frac{1}{n}\}$ and $F_n = \bigcap_{x \in E} C(x, n)$. It follows that $\{F_n\}$ is a decreasing sequence of nonempty, closed and convex sets. Since X has property (C), $E_C = \bigcap_{n=1}^{\infty} F_n$ is nonempty, closed and convex.

Definition 2.2. Let X be a metric space. A mapping $T : X \rightarrow X$ is said to satisfy condition (A) if for all $x, y \in X$ we have

$$d(Tx, Ty) \leq \max\{d(x, Tx), d(y, Ty), [d(x, Ty) + d(y, Tx)]/3, [d(x, y) + d(x, Tx) + d(y, Ty)]/3\},$$

The proof of the next theorem is similar to that used by Kirk [9] in Banach space. The theorem extends Kirk's theorem and Theorem 1 of Kannan [7].

Theorem 2.3. Let X be a convex metric space having property (C). Let D be a nonempty, bounded, closed and convex subset of X . Let $T : D \rightarrow D$ be a mapping satisfying condition (A) with the property

$$\sup_{z \in K} d(z, Tz) \leq \frac{\delta(K)}{2},$$

where K is any nonempty convex subset of D which is mapped into itself by T . Then T has a unique fixed point in D .

Proof. Let H be the family of all nonempty, closed and convex subsets of D which is mapped into itself by T . By property (C) and Zorn's Lemma it follows that H has a minimal element, say E .

By Lemma 2.1, E_C is nonempty, closed and convex. For $x \in E_C$ and $y \in E$ we have

$$\begin{aligned} d(Tx, Ty) &\leq \max \{ d(x, Tx), d(y, Ty), [d(x, Ty) + d(y, Tx)]/3, \\ &\quad [d(x, y) + d(x, Tx) + d(y, Ty)]/3 \} \\ &\leq \max \left\{ \frac{\delta(E)}{2}, [d(x, y) + d(y, Ty) + d(y, x) + d(x, Tx)]/3, \right. \\ &\quad \left. [d(x, y) + \delta(E)]/3 \right\} \\ &\leq \max \left\{ \frac{\delta(E)}{2}, [2d(x, y) + \delta(E)]/3, [d(x, y) + \delta(E)]/3 \right\} \\ &\leq \sup \{ d(x, y) : y \in E \} = R_x(E) = R(E). \end{aligned}$$

So, $T(E)$ is contained in a closed spherical ball

$S(Tx, R(E))$ and hence $T(E \cap \bar{S}) \subseteq E \cap \bar{S}$ (as $T(E) \subseteq E$). By the minimality of E , we conclude that $E \subseteq \bar{S}$.

Hence,

$$d(Tx, y) \leq R(E)$$

for all y in E . So,

$$(1) \sup \{ d(Tx, y) : y \in E \} \leq R(E).$$

Hence,

$$R_{T(x)}(E) = \sup \{ d(Tx, y) : y \in E \} \leq R(E),$$

which implies that

(2) $R_{T(x)}(E) = R(E)$, i. e. $T(x)$ belongs to E_C . Therefore, we have proved that $T(E_C) \subseteq E_C$.

We assert that, if E contains more than one element, then E_C is a proper subset of E . Suppose not, i. e. $E = E_C$. Then for x, y in E ,

$$R_x(E) = R_y(E) = R(E).$$

Therefore, $\sup \{d(x, z) : z \in E\} = \sup \{d(z, y) : z \in E\}$ for all x, y in E . From this it follows that $\sup \{d(x, z) : z \in E\} = R(E)$, a constant for all $x \in E$. Hence $\delta(E) = \sup \{d(x, z) : x, z \in E\} = R(E)$.

This implies that for $x \in E = E_C$ (and x in E_C)

$$(3) \quad \sup \{d(Tx, z) : z \in E\} = \delta(E).$$

For x, y in $E = E_C$, by the definition of T we have

$$d(Tx, Ty) \leq \max \{d(x, Tx), d(y, Ty), [d(x, Ty) + d(y, Tx)] / 3, [d(x, y) + d(x, Tx) + d(y, Ty)] / 3\}$$

$$\leq \max \left\{ \frac{\delta(E)}{2}, 2/3 \delta(E), [\delta(E) + \delta(E)] / 3 \right\}$$

$$\leq 2/3 \delta(E).$$

Therefore $T(E_C)$ is contained in a closed spherical ball $S(Tx, 2/3 \delta(E))$ and hence $T(E_C \cap \bar{S}) \subseteq E_C \cap \bar{S}$ (as $T(E_C) \subseteq E_C$). Then by the minimality of E , $E = E_C$, we conclude that $E_C \subseteq \bar{S}$. Hence $d(Tx, y) \leq 2/3 \delta(E)$ for all y in E . So, $\sup \{d(Tx, y) : y \in E\} \leq 2/3 \delta(E)$, which contradicts (3), because E contains more than one element. Thus, we conclude that if E contains more than one element, then E_C is a proper subset of E . But then in view of (2) it contradicts the minimality of E . Therefore, E contains only one element say v . Since T maps E into itself, v is a fixed point of T . The uniqueness of the fixed point v is obvious.

Theorem 1 [7] follows as a corollary of our theorem 2.3. Kirk [9] has proved a fixed point theorem using theorem of Smulian and the concept of normal structure where, however, the unicity is not guaranteed. The following example shows that our result is indeed an extension of Kirk's theorem,

Example 2.4. Let $X = [0, 7]$ with the usual metric. Define $T: X \rightarrow X$ as follows

$$Tx = \begin{cases} x/3 & 0 \leq x \leq 6 \\ -2x + 14 & 6 \leq x \leq 7 \end{cases}$$

Then T satisfies condition (A) with 0 only fixed point, but T is not nonexpansive. Indeed, let $x=6, y=7$ then $d(x, y) = 1 < 2 = d(Tx, Ty)$.

Definition 2.5. Let X be a convex metric space. We will say that X is *strictly convex* if for any x, y in X and $\lambda \in I$, there exists a unique $z \in X$ such that $\lambda d(x, y) = d(z, y)$ and $(1-\lambda)d(x, y) = d(z, x)$.

Lemma 2.6. Let X be a strictly convex metric space and C be a nonempty convex subset of X . Let $T: C \rightarrow C$ be a mapping as in Theorem 3.2, then $F(T)$ is convex.

Proof. Let x, y be in $F(T)$; we need to show that for any $\lambda, 0 \leq \lambda \leq 1$; $W(x, y; \lambda)$ belongs to $F(T)$. Since C is convex, $W(x, y; \lambda)$ belongs to C for x, y in $F(T)$ and $\lambda \in I$. Let $z = W(x, y; \lambda)$. Now

$$(1) \quad d(x, y) = d(Tx, Ty) \leq d(x, Tz) + d(Tz, y).$$

Using the definition of T we have

$$\begin{aligned} (2) \quad d(x, Tz) &\leq \max \{ d(x, z), [d(x, Tx) + d(z, Tz)]/2 \\ &\quad [d(x, Tz) + d(z, Tx)]/2 \} \\ &= \max \{ d(x, z), d(z, Tz)/2, [d(x, Tz) + d(z, x)]/2 \} \end{aligned}$$

Now $d(x, Tz) \leq 1/2 d(z, Tz) \leq 1/2 [d(z, x) + d(x, Tz)]$ implies $d(x, Tz) \leq d(x, z)$. Thus from (2) we have

$$(3) \quad d(x, Tz) = d(x, T(W(x, y; \lambda))) \leq d(x, z) = d(x, W(x, y; \lambda));$$

Similarly we have

$$(4) \quad d(y, T(W(x, y; \lambda))) \leq d(y, W(x, y; \lambda)),$$

Using (3) and (4) we can write (1) as

$$\begin{aligned} d(x, y) &\leq d(x, T(W(x, y; \lambda))) + d(T(W(x, y; \lambda)), y) \\ &\leq d(x, W(x, y; \lambda)) + d(W(x, y; \lambda), y) \\ &\leq \lambda d(x, x) + (1-\lambda) d(x, y) + \lambda d(y, x) + (1-\lambda) d(y, y) \\ &= d(x, y). \end{aligned}$$

Thus, we conclude that all inequalities are equalities,

i. e.

$$\begin{aligned} d(x, T(W(x, y; \lambda))) + d(T(W(x, y; \lambda)), y) \\ = d(x, W(x, y; \lambda)) + d(W(x, y; \lambda), y). \end{aligned}$$

Hence

$T(W(x, y, \lambda)) = W(x, y; \lambda)$. Therefore $F(T)$ is convex.

Definition 3. 1. Let X be a metric space $T: X \rightarrow X$ be

a mapping. T is said to have *property (K)* on $E \subset X$ if, for any closed subset F of E which contains more than one element and which is mapped into itself by T , there exists an $x \in F$ such that

$$d(x, Tx) < \sup_{y \in E} d(y, Ty).$$

Theorem 3. 2. Let X be a compact metric space and $T: X \rightarrow X$ be a continuous mapping satisfying the following condition :

$$\begin{aligned} d(Tx, Ty) &\leq \max \{ d(x, y), [d(x, Tx) + d(y, Ty)]/2, \\ &\quad [d(x, Ty) + d(y, Tx)]/2 \} \end{aligned}$$

for all $x, y \in X$. Suppose T has property (K) on X ; then T has a fixed point.

Proof. Let H be the family of all nonempty, closed subsets of X which is mapped into itself by T . By Zorn's Lemma it follows that H has a minimal element D . If D consists of a single element, then that element must be a fixed point of T . Suppose D contains more than one element. Since T has property (K) over X , there exists a $z \in D$ such that

$$d(z, Tz) = \gamma < \sup_{y \in D} d(y, Ty),$$

Let $C = \{x \in D : d(x, Tx) \leq \gamma\}$. Then C is nonempty proper subset of D . We claim C is closed too. Indeed, if $\{x_n\}$ is in C and $x_n \rightarrow y$, then $y \in D$ (since D is closed).

By the triangle inequality, $d(y, Ty) \leq (y, x_n) + d(x_n, Tx_n) + d(Tx_n, Ty)$. The continuity of T implies $d(y, Ty) \leq \gamma$, so y is in C . It remains to show that C is mapped into itself by T . Let $x \in C$, then

$$d(Tx, T^2x) \leq \max\{d(x, Tx), [d(x, Tx) + d(Tx, T^2x)]/2, [d(x, T^2x) + d(Tx, Tx)]/2\}$$

Hence $d(Tx, T^2x) \leq d(x, Tx) \leq \gamma$. So C is mapped into itself by T . Thus C is nonempty, closed proper subset of D which is mapped into itself. This contradicts the minimality of D . Hence D consists of a single point, which must be a fixed point of T .

The following examples show that our theorem is indeed a generalization of the theorem of Kannan [7].

Example 3.3. Let $X = \mathbb{R}$ with the usual metric. Define $T : X \rightarrow X$ by $Tx = x$ for all $x \in \mathbb{R}$. Then T satisfies condition of our mapping, but not that of Kannan.

Example 3. 4. Let $X = [0, \infty)$ with the usual metric and $Tx = \frac{x}{1+x}$. Then T satisfies condition of our mapping, but not that of Kannan. In fact, for $x \neq y$, $d(Tx, Ty) = \frac{(x-y)(xy-1)}{xy} < d(x, y)$, as T satisfies our condition. Let $x = 0, y = 1$, then $d(Tx, Ty) = 1/2 > 1/4 = 1/2 [d(x, Tx) + d(y, Ty)]$.

Theorem 3. 5. Let X be a strictly convex metric space with property (C). Let D be a nonempty, bounded, closed and convex subset of X . Let $F: D \rightarrow D$ be a family of continuous commuting mappings with nonempty fixed points set and satisfying condition of Theorem 3. 2. Then F has a common fixed point in D .

Proof: It follows from Lemma 2.6 that $F(T)$, the fixed points set of T in F , is convex. By the continuity of T we infer that $F(T)$ is closed.

Let F_α be the fixed point set of T_α in F ; F_α is nonempty, closed, bounded and convex. If $u \in F_\alpha$, then for any β , $T_\alpha T_\beta = T_\beta T_\alpha u = T_\beta u$; i.e. $T_\beta u$ lies in F_α and each T_β maps F_α into itself. If we are given a finite sequence $\alpha_1, \alpha_2, \dots, \alpha_m$, and consider T_{α_m} as a mapping satisfying condition of theorem of $F_{\alpha_1} \cap F_{\alpha_2} \cap \dots \cap F_{\alpha_m}$ into itself, it follows from the hypothesis that $\bigcap_{k=1}^m F_{\alpha_k} \neq \phi$. Since X has property (C), the family $\{F_\alpha\}$ has a nonempty intersection, but this consists of precisely the common fixed points.

Definition. Let B be a bounded set in a metric space X , and let $\delta(B)$ be its diameter. A point $x \in B$ is said to be a diametral point of B if $\sup_{y \in B} d(x, y) = \delta(B)$. A convex subset K of X is said to have

normal structure if every bounded convex subset K_1 of K which contains more than one point, has a point that is not a diametral point of K_1 .

Any compact convex metric space has normal structure [11 , Proposition 5]. Every compact convex set of Banach space has a normal structure [2 p. 6] ; so does every closed bounded and convex subset of a uniformly convex Banach space [10, p. 8] .

Corollary 3. 6. Suppose X is strictly convex with property (C). Let K be a nonempty, bounded, closed and convex subset of X with normal structure. If F is commuting family of nonexpansive mappings, of K into itself, then the family has a common fixed point.

As our final result, we prove an extension of the results of Goebel, Kirk and Shimi [6].

Definition 4. 1. Let X be a metric space, $T : X \rightarrow X$ be a mapping. T is said to have property (S) on $K \subset X$ if for any nonempty, closed bounded convex subset D of K which contains more than one element and which is mapped into itself by T , we have

$$\sup_{y \in D} d(y; Ty) < \delta(D),$$

where $\delta(D)$ is the diameter of D .

Definition 4. 2 Let X be a metric space and $T: X \rightarrow X$ be a mapping. We say that T satisfies condition (GKS) if for all $x, y \in X$ we have

$$\begin{aligned} d(Tx, Ty) \leq & a_1 d(x, y) + a_2 d(x, Tx) + a_3 d(y, Ty) \\ & + a_4 d(x, Ty) + a_5 d(y, Tx), \end{aligned}$$

where $a_1, a_4, a_5 \geq 0, a_2, a_3 > 0$ and $\sum_{i=1}^5 a_i = 1$.

Theorem 4.3. Let X be a convex metric space having property (C).

Let K be a nonempty closed, bounded convex subset of X . Let $T: K \rightarrow K$ be a mapping satisfying condition (GKS) and having property (S) on X . Then T has a unique fixed point.

Proof : Let F be the family of all nonempty closed bounded convex subsets of K which are mapped into itself. Since $K \in F$, $F \neq \Phi$, using property (C) and Zorn's Lemma we can find a minimal subset D of K in F . Without loss of generality we may assume that $a_2 = a_3$ and $a_4 = a_5$. Let

$$\gamma(D) = (a_1 + a_4 + a_5) \sup_{x, y \in D} d(x, y) + (a_2 + a_3) \sup_{y \in D} d(y, Ty).$$

Then $\gamma(D) < (a_1 + 2a_4) \delta(D) + 2a_2 \delta(D) \leq \delta(D)$, since $a_2 > 0$. We claim that D contains one point. Suppose not, let $x, y \in D$, $x \neq y$.

Then,

$$\begin{aligned} d(Tx, Ty) &\leq a_1 d(x, y) + a_2 d(x, Tx) + a_3 d(y, Ty) \\ &\quad + a_4 d(x, Ty) + a_5 d(y, Tx) \\ &\leq (a_1 + 2a_4) \sup_{x, y \in D} d(x, y) + 2a_2 \sup_{y \in D} d(y, Ty) \\ &= \gamma(D). \end{aligned}$$

Thus $T(D)$ is contained in a closed spherical ball C with center Tx and radius $\gamma(D)$.

It is obvious that $T(D \cap C) \subseteq D \cap C$. By the minimality of D we obtain $D \subseteq C$. Therefore $d(Tx, y) \leq \gamma(D)$ for each $y \in D$ and hence

$\sup_{y \in D} d(Tx, y) \leq \gamma(D)$ for any arbitrary but fixed $x \in D$. Let

$$D_1 = \{z \in D : \sup_{y \in D} d(z, y) \leq \gamma(D)\}.$$

Then D_1 is nonempty, bounded closed and convex. Moreover

$T(D_1) \subset D_1$. Also,

$$\delta(D_1) = \sup_{x, y \in D_1} d(x, y) \leq \sup_{\substack{y \in D \\ x \in D_1}} d(x, y)$$

$$\leq \sup_{x \in D_1} \sup_{y \in D} d(x, y) \leq \gamma(D)$$

$$< \delta(D).$$

Thus, D_1 is a proper subset of D . This contradicts the minimality of D . Consequently D has only one point which is a fixed point of T in K . The unicity of fixed point is clear.

Remark 4. 4. A similar theorem for uniformly convex Banach was proved by Goebel, Kirk, and Shimi [16] without the restriction that $a_2, a_3 > 0$, however there the continuity of the mapping T was assumed; moreover the fixed point was not unique,

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FIXED POINTS FOR MULTI - VALUED MAPPINGS IN LOCALLY CONVEX SPACE

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ABSTRACT

Let $\mathcal{Q}$ be a family of continuous seminorms determining the topology of a locally convex topological vector space E . In this paper, sufficient conditions are given for the existence of fixed points for multivalued mappings. The results of this paper strengthen those of Kirk, Assad and Kirk Markin, and others. Our results are motivated by the results of Samanta (12) and extend his results to locally convex spaces.

1. INTRODUCTION

Let X be a metric space and $(2^X, H)$ be the space of all nonempty compact subsets of X endowed with the Hausdorff metric H . Let $T: X \rightarrow 2^X$ be a multivalued mapping. A point $x \in X$ is said to be a fixed point of T , provided $x \in T(x)$. Kirk [3] proved the following:

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If K is a nonempty, closed, bounded, and convex subset of a reflexive Banach space X and if K possesses normal structure, then every self-nonexpansive mapping of K has a fixed point. Markin [8] extended the result of Kirk to multivalued mappings. Markin's result is : Let X be a real Hilbert space and $T : X \rightarrow 2^X$ be nonexpansive. If

$T(x) \subset B$ for $x \in B$, where $B = \{x \in X; \|x\| \leq 1\}$,

then T has a fixed point.

Assad and Kirk [1] have extended this result of Markin to Banach spaces satisfying Opial's condition. Fixed points for set-valued nonexpansive mappings and for set-valued Kannan-type mappings in locally convex spaces were given by Su and Sehgal [11] and Ko and Tsai [5], respectively. Our results, which are primarily motivated by the work of Ko [6], Ko and Tsai [5], and Su and Sehgal [11], extend the results of Assad and Kirk [1], Kirk [4], Markin [8], Nadler [9], and others.

2. PRELIMINARIES AND NOTATIONS

Throughout this paper, E will denote a Hausdorff, locally convex, linear topological space and Q , a (fixed) family of continuous seminorms which generate the topology of E . For any $p \in Q$ and $A \subseteq E$, $\delta_p(A)$ will denote the p -diameter of A , that is, $\delta_p(A) = \sup \{p(x-y) : x, y \in A\}$, $B_p(x, r)$ ($B_p[x, r]$) will denote the open (closed) ball with center at x and radius r . For $A \subseteq E$, $d_p(x, A) = \inf \{p(x-y) : y \in A\}$. We shall denote by 2^E , the space of all nonempty compact subsets of E .

Definition 2.1 A convex subset K of E is said to have *normal structure* if for every closed, bounded, convex subset W of K such that $\delta_p(W) > 0$, there exists $x \in W$ such that $\sup \{p(x-y) : y \in W\} < \delta_p(W)$.

Definition 2.2 Let K be a nonempty subset of E , and let $\{W_\alpha : \alpha \in \Delta\}$ be a

decreasing set of nonempty bounded subsets of K . For $x \in K$, $\alpha \in \Delta$, and $p \in Q$, define

$$r_{p,\alpha}(x) = \sup \{ p(x - y) : y \in W_\alpha \}$$

$$r_p(x) = \inf \{ r_{p,\beta}(x) : \beta \in \Delta \}$$

$$r_p = \inf \{ r_p(x) : x \in K \}$$

$$M = \{ x \in K : r_p(x) = r_p \}.$$

The set M and the number r_p will be called the asymptotic center and the asymptotic radii, respectively, of $\{ W_\alpha : \alpha \in \Delta \}$ in K with respect to p .

Definition 2.3 Let K be a nonempty subset of E .

$T: K \rightarrow 2^K$ is said to be a *nonexpansive map* if for each $p \in Q$, for each $x, y \in K$, and for each $x_1 \in T(x)$, there exists $y_1 \in T(y)$ such that $p(x_1 - y_1) \leq p(x - y)$.

Note that if the set K is compact, then each set A in 2^K is compact and, hence, bounded under each $p \in Q$.

Thus, the Hausdorff metric D_p [2] induced from the seminorm p is well defined on 2^K . It should be noted that D_p is a metric on 2^K , although p is a seminorm on K . Hence, if K is compact, Definition 2.3 is equivalent to the following:

Definition 2.4 Let K be a compact subset of E . A map $T: K \rightarrow 2^K$ is said to be a nonexpansive map if for each $p \in Q$ and $x, y \in K$, we have $D_p(Tx, Ty) \leq p(x - y)$.

3. RESULTS

Theorem 3.1 Let E be a semireflexive, locally convex Hausdorff topological vector space and K be a closed, bounded, convex subset of E , possessing normal structure. If $T: K \rightarrow 2^K$ is a mapping such that

- (i) $T(x) \cap K = \phi$ for all $x \in K$,
- (ii) for any closed convex subset L of K satisfying

$T(w) \cap L = \phi$ for all $w \in L$,

$D_p(T(x) \cap L, T(y) \cap L) \leq p(x-y)$ whenever

$x, y \in L, x \neq y$. Then T has a fixed point.

Proof Let G be the family of all nonempty, closed convex subsets M of K such that $T(x) \cap M \neq \phi$ for all $x \in M$. Then, G is nonempty, since $K \in G$. Partially order G by inclusion. Let $F = \{F_i\}_{i \in \Delta}$

be a decreasing chain in G . It follows that $\bigcap_{i \in \Delta} F_i \neq \phi$

(due to the semireflexivity of E). Let $F_0 = \bigcap_{i \in \Delta} F_i$,

a nonempty, closed convex subset of M . Let $x \in F_0$. By hypothesis, $T(x) \cap (\bigcap_{i \in \Delta} F_i) \neq \phi$. For any

$F_i \in F, T(x) \cap F_i$ is compact and

$\{T(x) \cap F_i\}_{i \geq 1}$ is a family of nonempty

closed subsets of the compact set $T(x) \cap F_i$ having finite intersection property. Consequently, $\bigcap_{i \geq 1} (T(x) \cap F_i) \neq \phi$

and, therefore, $\bigcap_{i \in \Delta} (T(x) \cap F_i) \neq \phi$,

i. e., $T(x) \cap F_0 \neq \phi$. Thus, any chain in G has a greatest lowerbound, and by Zorn's Lemma, there is a minimal member N in G .

We claim that N is a singleton set. If not, then as shown by Lim [7, p 315-316], the center of N , denoted by R is a nonempty proper closed convex subset of N . It is enough to show that R belongs to G . Let $y \in R$, then $y \in N$ and

$T(y) \cap N \neq \phi$. Take $z \in T(y) \cap N$. Let $W = B_p[z, r(N)] \cap N$

Then, W is a nonempty closed convex subset of N . Let $x \in W$.

Using Hypothesis (ii), we have $D_p(T(x) \cap N, T(y) \cap N) \leq p(x-y)$.

Since $z \in T(y) \cap N$, there is a $u \in T(y) \cap N$ such that

$$p(u-y) \leq D_p(T(x) \cap N, T(y) \cap N) \leq p(x-y). \text{ Hence,}$$

$T(x) \cap W \neq \phi$ for all $x \in W$. By the minimality of N , we have

$N \subseteq W$, that is, $N \subseteq B_p[z, r(N)]$. Consequently, $z \in R$ and

we have $T(y) \cap R \neq \phi$ for all $y \in R$. Therefore $R \in G$, a contradiction to the minimality of N . Hence, N consists of a single point which is a fixed point of T .

The following example shows that Condition (ii) cannot be replaced by $D_p(Tx, Ty) \leq p(x-y)$ for all

$x, y \in K, x \neq y$.

Example 2.1 Let $X = R^2$ with usual norm and

$K = \{(x, 0) : 0 \leq x \leq 1\}$. Then, K is a bounded closed convex

subset of a reflexive Banach space X . Define $T: K \rightarrow 2^X$ as follows:

$$T(x, 0) = \left\{ \left(\frac{1}{\sqrt{2}} \left(\frac{1}{2} - x \right), \sqrt{\frac{1}{2}} \left(\frac{1}{2} - x \right) \right), \left(x + \frac{1}{2}, 0 \right) \right\} \quad 0 \leq x \leq \frac{1}{2},$$

$$T\left(x + \frac{1}{2}, 0\right) = \left\{ \left(1 - \frac{x}{\sqrt{2}}, \sqrt{\frac{x}{2}} \right), (x, 0) \right\}, \quad 0 \leq x \leq \frac{1}{2}.$$

Then $T(x)$ intersects K for every x in K . If

$(x, 0), (y, 0) \in K$ such that $x, y \in [0, \frac{1}{2}]$ or $x, y \in [\frac{1}{2}, 1]$,

then $D(T(x, 0), T(y, 0)) \leq \|(x, 0) - (y, 0)\|$. If

$x, y \in [0, \frac{1}{2}]$, then $D(T(x, 0), T(y + \frac{1}{2}, 0))$

$$\leq \left[\left(\frac{1}{\sqrt{2}} \left(\frac{1}{2} - x \right) - y \right)^2 + \left(\sqrt{\frac{1}{2}} \left(\frac{1}{2} - x \right) \right)^2 \right]^{1/2}$$

$$= [x^2 + y^2 + \sqrt{2}xy - (x + \sqrt{\frac{y}{2}} + \frac{1}{4})]^{1/2},$$

whenever $(\sqrt{2} + 2)xy \leq (1 + \frac{1}{\sqrt{2}})y$, i.e., $x \leq \frac{1}{2}$ with $y \neq 0$.

Thus, $D(T(x, 0), T(y, 0)) \leq \|(x, 0) - (y, 0)\|$ for all $(x, 0), (y, 0)$, $x \neq y$ in K . Now for $x, y \in [0, \frac{1}{2}]$,

$$\begin{aligned} D(T(x, 0) \cap K, T(y + \tfrac{1}{2}, 0) \cap K) &= \|(x + \tfrac{1}{2}, 0) - (y, 0)\| \\ &= \|x + \tfrac{1}{2} - y\| \\ &= \|y + \tfrac{1}{2} - x\| \end{aligned}$$

whenever $x > y$. So T is nonexpansive and satisfies all but Condition (ii) of Theorem 2.1. Clearly, T has no fixed point.

For the proof of Theorem 2.2, we need to recall the following :

Lemma 2.1 [10, Lemma 3.1]. Let H and K be two closed subsets of E such that $H \cap K \neq \phi$. If H is convex, then

$\partial_H(K) = \phi$ if and only if $H \subseteq K$ (here,

$\partial_H(K)$ denotes the relative boundary of $H \cap K$ in H).

Theorem 2.2 Let E be a Hausdorff, locally convex topological vector space and K be a nonempty, closed convex subset of E . Let L be a nonempty, weakly compact, convex subset of K possessing normal structure and $T : L \rightarrow 2^E$

be a mapping satisfying $T(x) \cap K \neq \phi$ for all $x \in L$. Further, for M a nonempty, closed convex subset of K such that $T(x) \cap M \neq \phi$, whenever $x \in M \cap L$, the following conditions hold :

(i) $D_p(T(x) \cap M, T(y) \cap M) \leq p\|x - y\|$ for all $x, y \in M \cap L$, for all $p \in Q$, and

(ii) $T(W) \cap (M \cap L) \neq \phi$, where $w \in \partial_M(L)$.

Then, T has a fixed point.

Proof Let G be the family of all nonempty closed convex subsets M of K such that $M \cap L \neq \phi$ and $T(x) \cap M \neq \phi$ whenever $x \in M \cap L$. Then G is nonempty, since $K \in G$. Partially order G by inclusion. Arguing similarly, as in the proof of Theorem 2.1, it follows that every chain in G has a greatest lower bound. So, by Zorn's Lemma, there is a minimal element N in G . If $\partial_N(L) = \phi$, then, by Lemma 2.1, $N \subseteq L$. In this case, Theorem 2.1 is applicable, and T has a fixed point. Thus, we assume that $\partial_N(L) \neq \phi$ and show that $N \cap L$ is a singleton set. Let $\delta_P(N \cap L) > 0$, for some p .

Since L has normal structure, there is a point $y \in L \cap N$ such that

$$0 < \alpha = \sup_{x \in N \cap L} p(x - y) < \delta(N \cap L). \quad (1)$$

Consider the set $H = \{x \in N : N \cap L \subseteq B_P[x, \alpha]\}$, a closed convex subset of N . Since $y \in H$, H is nonempty and, by (1) $H \neq N$. We show that $H \in G$. Let $z \in H \cap L$ and

$y \in \partial_N(L)$. Then, $y \in N \cap L$ and, hence,

$p(z - y) \leq \alpha$. Now, since $N \in G$ from Hypothesis (i), we have.

for $y, z \in N \cap L$,

$$D_P(T(y) \cap N, T(z) \cap N) \leq p(y - z). \quad (2)$$

Consequently, $D_P(T(y) \cap N, T(z) \cap N) \leq \alpha$. Since $y \in \partial_N(L)$, by Hypothesis (ii), $T(y) \cap (N \cap L) \neq \phi$.

Take $u \in T(y) \cap (N \cap L)$. Then, using (2), we obtain for

$$v \in T(z) \cap N, p(u - v) \leq D_P(T(z) \cap N, T(y) \cap N) \leq p(z - y).$$

Let $W = B_P[v, \alpha] \cap N$. Then, W is a closed convex subset of K with $W \cap L \neq \phi$ (since $u \in W \cap L$). If $w \in W \cap L \cap N \cap L$, then $p(z - w) \leq \alpha$.

Further, $D_p(T(W) \cap N, T(z) \cap N) \leq p(z-w) \leq \alpha$.

Since $v \in T(z) \cap N$, there exists an element $s \in T(w) \cap N$ such that $p(v-s) \leq D_p(T(w) \cap N, T(z) \cap N) \leq p(w-z) \leq \alpha$.

This shows that $s \in W$, or that $T(w) \cap W \neq \phi$ for all

$w \in W \cap L$. Therefore, $W \in G$. By the minimality of N , we have $N = W$. Hence, $N \cap L \subseteq B_p[v, \alpha]$ and $v \in H$.

We now have $T(z) \cap H \neq \phi$ for all $z \in H \cap L$, and it follows that $H \in G$. But H is a proper subset of N , a contradiction. Hence, $N \cap L$ is a singleton set. Let $N \cap L = \{x\}$.

By assumption, $\phi \neq \partial_N(L \subseteq N \cap L, \partial_N(L) = \{x\})$.

Now, by Hypothesis (ii), $x \in T(x)$. This completes the proof.

Remark 2.1 The example below shows that Condition (ii) of Theorem 2.2 cannot be replaced by the following condition :

(ii)' If $x \in \partial_K(T)$, then $T(x) \cap T \neq \phi$.

Example 2.2 Let $X = R^2$ with the usual norm and

$K = \{(x, 0) : 0 \leq x < \infty\}$ and

$L = \{(x, 0) : 0 \leq x \leq 1\}$. Define $T : L \rightarrow 2^X$

by $T(x, 0) = \{(1+x, 0), (0, 1-x)\}, (x, 0) \in L$.

Then K is a closed and convex subset of a reflexive Banach space, and L is a closed, bounded convex subset of K . The mapping T satisfies the condition $T(x) \cap K \neq \phi$ for all $x \in L$, Condition (i) of Theorem 2.2 and Condition (ii)' are also satisfied. By taking $M = [1/2, \infty)$, we see that Hypothesis (ii) of Theorem 2.2 is not satisfied. Clearly, T does not have a fixed point.

Remark 2.2 Our Theorem 2.2 is applicable to $L_p[0, 1]$, $1 < p < 2$. However, the results of Markin, and Assad and Kirk, do not hold there, since $L_p[0, 1]$ is not a Hilbert space, nor does it satisfy Opial's condition.

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ON SOME FINITE INTEGRALS INVOLVING THE MULTIVARIABLE H - FUNCTION

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ABSTRACT

In this paper, we evaluate some finite integrals involving the products of Jacobi polynomials and the H -function of several complex variables .

1. INTRODUCTION

Srivastava and Panda [13] have defined the multivariable H -function as (see also [12] and [14])

$$\begin{aligned}
 H \left(\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix} \right) &= H_{\substack{O, \varepsilon : (M', N') ; \dots ; (M^{(r)}, N^{(r)}) \\ A, C : (B', D') ; \dots ; (B^{(r)}, D^{(r)})}} \left(\begin{matrix} [(a) : \theta', \dots, \theta^{(r)}] : \\ [(c) : C', \dots, C^{(r)}] : \\ [(b') : \phi'] ; \dots ; [(b^{(r)}) : \phi^{(r)}] ; z_1 \\ \vdots \\ [(d') : \delta'] ; \dots ; [(d^{(r)}) : \delta^{(r)}] ; z_r \end{matrix} \right) \\
 &= \frac{1}{(2\pi i)^r} \int_{L_1} \dots \int_{L_r} V(s_1, \dots, s_r) U_1(s_1) \dots U_r(s_r) z_1^{s_1} \dots z_r^{s_r} \\
 &\quad \cdot ds_1 \dots ds_r \dots (1.1)
 \end{aligned}$$

where

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$$V(s_1, \dots, s_r) = \frac{\prod_{j=1}^{\varepsilon} \Gamma(1 - aj + \sum_{i=1}^r \theta_j^{(i)} s_i)}{\prod_{j=\varepsilon+1}^A \Gamma(a_j - \sum_{i=1}^r \theta_j^{(i)} s_i) \prod_{j=1}^C \Gamma(1 - c_j + \sum_{i=1}^r C_j^{(i)} s_i)} \quad \dots (1.2)$$

and

$$U_i(s_i) = \frac{\prod_{j=1}^{M^{(i)}} \Gamma(d_j^{(i)} - \delta_j^{(i)} s_i) \prod_{j=1}^{N^{(i)}} \Gamma(1 - b_j^{(i)} + \phi_j^{(i)} s_i)}{\prod_{j=M^{(i)}+1}^{D^{(i)}} \Gamma(1 - d_j^{(i)} + \delta_j^{(i)} s_i) \prod_{j=N^{(i)}+1}^{B^{(i)}} \Gamma(b_j^{(i)} - \phi_j^{(i)} s_i)} \quad i = 1, \dots, r \quad \dots (1.3)$$

The multiple integral in (1.1) absolutely converges if

$$|\arg(z_i)| < \frac{1}{2} T_i \pi, \quad i = 1, \dots, r \quad \dots (1.4)$$

where

$$T_i = - \sum_{j=\varepsilon+1}^A \theta_j^{(i)} + \sum_{j=1}^{N^{(i)}} \phi_j^{(i)} - \sum_{j=N^{(i)}+1}^{B^{(i)}} \phi_j^{(i)} - \sum_{j=1}^C C_j^{(i)} + \sum_{j=1}^{M^{(i)}} \delta_j^{(i)} - \sum_{j=M^{(i)}+1}^{D^{(i)}} \delta_j^{(i)} > 0, \quad i = 1, \dots, r \quad \dots (1.5)$$

The following notations and known results will be used throughout this paper :

$$(\alpha)_n = \frac{\Gamma(\alpha + n)}{\Gamma(\alpha)} = \begin{cases} 1, & \text{if } n = 0, \\ \alpha(\alpha+1) \dots (\alpha+n-1), & \forall n \in \{1, 2, 3, \dots\}. \end{cases} \quad \dots (1.6)$$

Next for the Jacobi polynomials $P_n^{(\alpha, \beta)}(x)$ [11, p. 254, Eq.(1)], we have

$$P_k^{(\alpha, \beta)}(t+\rho) P_k^{(\alpha, \beta)}(t-\rho) = \frac{(-1)^k (1+\alpha)_k (1+\beta)_k}{(k!)^2}$$

$$\sum_{r=0}^k \frac{(-k)_r (1+\alpha+\beta+k)_r}{(1+\alpha)_r (1+\beta)_r} P_r^{(\alpha, \beta)}(x) t^r \quad \dots (1.7)$$

$$\rho^k P_k^{(\alpha, \alpha)}\left(\frac{1-xt}{\rho}\right) = \frac{(1+\alpha)_k}{k!} \sum_{r=0}^k \frac{(-k)_r}{(1+\alpha)_r} P_r^{(\alpha, \alpha)}(x) t^r \quad \dots (1.8)$$

$$\frac{1}{\rho} (1-t+\rho)^{-\alpha} (1+t+\rho)^{-\beta} = 2^{-\alpha-\beta} \sum_{r=0}^{\infty} P_r^{(\alpha, \beta)}(x) t^r \quad \dots (1.9)$$

In each of the formulas (1.7), (1.8), and (1.9), and throughout this paper, $\rho = (1-2xt + t^2)^{1/2}$. Formulas (1.7), (1.7) and (1.9) can be found, for example in [4, p. 945], [4, p. 946] and [6, p. 172], respectively.

$$\int_{-1}^1 (1+x)^{\alpha} (1-x)^{\beta} P_n^{(\alpha, \beta)}(x) H \left(\begin{matrix} z_1 (1+x)^{h_1} \\ \vdots \\ z_r (1+x)^{h_r} \end{matrix} \right) dx$$

$$= \frac{2^{\alpha+\beta+1} \Gamma(\alpha+n+1)}{n!} H_{\substack{0, \varepsilon+2 : (M', N') ; \dots ; (M^{(r)}, N^{(r)}) \\ A+2, C+2 : (B', D') ; \dots ; (B^{(r)}, D^{(r)})}}$$

$$\left(\begin{matrix} [(a) : \theta', \dots, \theta^{(r)}], (-\eta : h_1, \dots, h_r), (\beta-\eta : h_1, \dots, h_r) : \\ [(c) : C', \dots, C^{(r)}], (-\alpha-\eta-n-1 : h_1, \dots, h_r), (\beta-\eta+n : h_1, \dots, h_r) : \end{matrix} \right.$$

$$\left. \begin{matrix} [(b') : \phi'] ; \dots ; [(b^{(r)}) : \phi^{(r)}] ; 2^{h_1} z_1 \\ [(d') : \delta'] ; \dots ; [(d^{(r)}) : \delta^{(r)}] ; 2^{h_r} z_r \end{matrix} \right) \quad \dots (1.10)$$

Formula (1. 10) was given by Srivastava and Panda [13, p. 131, Eq. (2.2)] .

$$\int_{-1}^1 (1-x)^{\eta} (1+x)^{\sigma} P_n^{(\alpha, \beta)}(x) H \left(\begin{matrix} z_1 (1-x)^{h_1} (1+x)^{k_1} \\ z_r (1-x)^{h_r} (1+x)^{k_r} \end{matrix} \right) dx$$

$$= 2^{\eta+\sigma+1} \sum_{m=0}^n \frac{(-n)_m (\alpha+\beta+n+1)_m}{(\alpha+1)_m m!}$$

$$\begin{aligned} & 0, \epsilon+2 : (M^j, N') ; \dots ; (M^{(r)}, N^{(r)}) \\ & H \\ & A+2, C+1 : (B', D') ; \dots ; (B^{(r)}, D^{(r)}) \end{aligned}$$

$$\left(\begin{aligned} & [(a) : \theta', \dots, \theta^{(r)}], (-\sigma : k_1, \dots, k_r), (-m-\eta : h_1, \dots, h_r) : \\ & [(c) : C', \dots, C^{(r)}], (-1-m-\eta-\sigma : h_1 + k_1, \dots, h_r + k_r) : \end{aligned} \right)$$

$$\left(\begin{aligned} & [(b') : \phi'] ; \dots ; [(b^{(r)}) : \phi^{(r)}] ; z_1 2^{h_1+k_1} \\ & [(d') : \delta'] ; \dots ; [(d^{(r)}) : \delta^{(r)}] ; z_r 2^{h_r+k_r} \end{aligned} \right) \dots (1.11)$$

The integral (1.11) can be evaluated by making use of a known result [7, p. 284, Fq. (2)] . In fact, (1.12) is an obvious special case of much more general results given by Srivastava and Singh [15, p. 172, Fq. (3.2)] .

2. MAIN INTEGRALS

The following integrals have been evaluated in this paper :

$$\int_{-1}^1 (1+x)^{\eta} (1-x)^{\alpha} P_k^{(\alpha, \beta)}(t+\rho) P_k^{(\alpha, \beta)}(t-\rho) H \left(\begin{matrix} z_1 (1+x)^{h_1} \\ z_r (1+x)^{h_r} \end{matrix} \right) dx$$

$$= 2^{\eta+\alpha+1} \frac{(-1)^k \Gamma(1+\alpha+k) (1+\beta)_k}{(k!)^2}$$

$$\sum_{R=0}^k \frac{(-k)_R (1+\alpha+\beta+k)_R}{(1+\beta)_R} \frac{t^R}{R!}$$

$${}^H_{0, \epsilon+2 : (M', N') ; \dots ; (M^{(r)}, N^{(r)})} \\ A+2, C+2 : (B', D') ; \dots, (B^{(r)}, D^{(r)})$$

$$\left(\begin{aligned} & [(a) : \theta', \dots, \theta^{(r)}], (-\eta : h_1, \dots, h_r), (\beta-\eta : h_1, \dots, h_r) : \\ & [(c) : C', \dots, C^{(r)}], (-\alpha-\eta-R-1 : h_1, \dots, h_r), (\beta-\eta+R : h_1, \dots, h_r) : \\ & [(b') : \phi'] ; \dots ; [(b^{(r)}) : \phi^{(r)}] ; z_1 2^{h_1} \\ & [(d') : \delta'] ; \dots ; [(d^{(r)}) : \delta^{(r)}] ; z_r 2^{h_r} \end{aligned} \right) \dots (2.1)$$

where

$$\operatorname{Re}(\eta) > -1, \operatorname{Re}(\alpha) > -1,$$

$$h_i > 0, i = 1, \dots, r \text{ and } |\arg(z_i)| < \frac{1}{2} T_i \pi, i = 1, \dots, r,$$

$$T_i > 0 \text{ and } \operatorname{Re}(\eta + \sum_{i=1}^r h_i d_j^{(i)} / \delta_j^{(i)}) > -1.$$

$$\int_{-1}^1 \rho^k (1-x)^\eta (1-x)^\alpha P_k^{(\alpha, \alpha)} \left(\frac{1-xt}{\rho} \right) H \left(\begin{matrix} z_1 (1+x)^{h_1} \\ \vdots \\ z_r (1+x)^{h_r} \end{matrix} \right) dx \\ = \frac{2^{\eta+\alpha+1} \Gamma(1+\alpha+k)}{k!} \sum_{R=0}^k \frac{(-k)_R}{R!} t^R$$

$${}^H_{0, \epsilon+2 : (M', N') ; \dots ; (M^{(r)}, N^{(r)})}$$

$$A+2, C+2 : (B', D') ; \dots ; (B^{(r)}, D^{(r)})$$

$$\left(\begin{aligned} & [(a) : \theta', \dots, \theta^{(r)}], (-\eta : h_1, \dots, h_r), (\alpha-\eta : h_1, \dots, h_r) : \\ & [(c) : C', \dots, C^{(r)}], (-\alpha-\eta-R-1 : h_1, \dots, h_r), (\alpha-\eta+R : h_1, \dots, h_r) : \end{aligned} \right)$$

$$\left[(b') : \phi' \right] ; \dots ; \left[(b^{(r)}) : \phi^{(r)} \right] ; z_1 2^{h_1} \begin{matrix} \vdots \\ z_r 2^{h_r} \end{matrix} \left[(d') : \delta' \right] ; \dots ; \left[(d^{(r)}) : \delta^{(r)} \right] ; \dots (2.2)$$

where,

$$\operatorname{Re}(\eta) > -1, \operatorname{Re}(\alpha) > -1, h_i > 0, i = 1, \dots, r,$$

$$|\arg(z_i)| < \frac{1}{2} T_i \pi, T_i > 0 \text{ and}$$

$$\operatorname{Re}(\eta + \sum_{i=1}^r h_i \frac{d_j^{(i)}}{\delta_j^{(i)}}) > -1, i = 1, \dots, r$$

$$\int_{-1}^1 \frac{1}{\rho} (1-t+\rho)^{-\alpha} (1+t+\rho)^{-\beta} (1+x)^{\alpha} (1-x)^{\beta} H \begin{pmatrix} z_1 (1+x)^{h_1} \\ \vdots \\ z_r (1+x)^{h_r} \end{pmatrix} dx$$

$$= 2^{\alpha+\beta+1} \sum_{R=0}^{\infty} \frac{\Gamma(\alpha+R+1)}{R!} t^R H \begin{matrix} 0, \alpha+2; (M', N); \dots; (M^{(r)}, N^{(r)}) \\ A+\frac{1}{2}, C+2; (B', D'); \dots; (B^{(r)}, D^{(r)}) \end{matrix}$$

$$\left(\left[(a) : \theta', \dots, \theta^{(r)} \right], (-\eta : h_1, \dots, h_r), (\beta - \eta : h_1, \dots, h_r) ; \right. \\ \left. \left[(c) : C', \dots, C^{(r)} \right], (-\alpha - \eta - R - 1 : e_1, \dots, h_r), (\beta - \eta + R : h_1, \dots, h_r) ; \right.$$

$$\left. \left[(b') : \phi' \right] ; \dots ; \left[(b^{(r)}) : \phi^{(r)} \right] ; z_1 2^{h_1} \begin{matrix} \vdots \\ z_r 2^{h_r} \end{matrix} \left[(d') : \delta' \right] ; \dots ; \left[(d^{(r)}) : \delta^{(r)} \right] ; \dots (2.3) \right.$$

where

$$\operatorname{Re}(\eta) > -1, \operatorname{Re}(\alpha) > -1, h_i > 0,$$

$$|\arg(z_i)| < \frac{1}{2} T_i \pi, T_i > 0 \text{ and}$$

$$\operatorname{Re}(\eta + \sum_{i=1}^r h_i d_j^{(i)} / \delta_j^{(i)}) > -1, i = 1, \dots, r$$

$$\int_{-1}^1 (1-x)^{\alpha} (1+x)^{\beta} P_k^{(\alpha, \beta)}(t+\rho) P_k^{(\alpha, \beta)}(t-\rho) H \left(\begin{matrix} z_1 (1-x)^{h_1} (1+x)^{k_1} \\ \vdots \\ z_r (1-x)^{h_r} (1+x)^{k_r} \end{matrix} \right) dx$$

$$= 2^{\alpha+\beta+1} \frac{(-1)^k \Gamma(1+\alpha+k) \Gamma(1+\beta+k)}{(k!)^2}$$

$$\sum_{R=0}^k \sum_{m=0}^R \frac{(-k)_r (1+\alpha+\beta+k)_R}{\Gamma(1+\alpha+R) \Gamma(1+\beta+R)} t^R \frac{(-R)_m (\alpha+\beta+R+1)_m}{(\alpha+1)_m m!}$$

$$0, \epsilon+2 : (M', N') ; \dots ; (M^{(r)}, N^{(r)})$$

$$H \quad A+2, C+1 : (B', D) ; \dots ; (B^{(r)}, D^{(r)})$$

$$[(a) : \theta', \dots, \theta^{(r)}], (-\sigma : k_1, \dots, k_r), (m-\eta : h_1, \dots, h_r) :$$

$$[(c) : C', \dots, C^{(r)}], (-1-m-\eta-\sigma : h_1+k_1, \dots, h_r+k_r) :$$

$$\left[(b) : \phi' ; \dots ; [(b^{(r)}) : \phi^{(r)}] ; z_1 2^{h_1+k_1} \right. \\ \left. [(d') : \delta'] ; \dots ; [(d^{(r)}) : \delta^{(r)}] ; z_r 2^{h_r+k_r} \right) \dots (2.4)$$

where

$$Re(\gamma_i) < -1, Re(\sigma) < -1, k_i > 0,$$

$$h_i > 0, |arg(z_i)| < \frac{1}{2} T.\pi \text{ and}$$

$$Re(\eta + \sum_{i=1}^r h_i d_j^{(i)} / \delta_j^{(i)}) > -1, Re(\sigma + \sum_{i=1}^r k_i d_j^{(i)} / \delta_j^{(i)}) > -1,$$

$$T_i > 0, i = 1, \dots, r$$

$$\int_{-1}^1 (1-x)^{\alpha} (1+x)^{\beta} \rho^k P_k^{(\alpha, \beta)} \left(\frac{1-xt}{\rho} \right) H \left(\begin{matrix} z_1 (1-x)^{h_1} (1+x)^{k_1} \\ \vdots \\ z_r (1-x)^{h_r} (1+x)^{k_r} \end{matrix} \right) dx$$

$$= \frac{2^{\alpha+1} \Gamma(1+\alpha+k)}{k!} \sum_{R=0}^k \sum_{m=0}^r \frac{(-k)_R}{\Gamma(1+\alpha+R)} t^R \frac{(-R)_m (2\alpha+R+1)_m}{(\alpha+1)_m m!}$$

$$\begin{aligned} & 0, \varepsilon+2 : (M', N') ; \dots ; (M^{(r)}, N^{(r)}) \\ \cdot H & A+2, C+1 : (B', D') ; \dots ; (B^{(r)}, D^{(r)}) \end{aligned}$$

$$\left(\begin{aligned} & [(a) : \theta', \dots, \theta^{(r)}], (-\sigma : k_1, \dots, k_r), (-m-\eta : h_1, \dots, h_r) : \\ & [(c) : C', \dots, C^{(r)}], (-1-m-\eta-\sigma : h_1+k_1, \dots, h_r+k_r) : \end{aligned} \right.$$

$$\begin{aligned} & [(b') : \phi'] ; \dots ; [(b^{(r)}) : \phi^{(r)}] ; z_1 2^{h_1+k_1} \\ & [(d') : \delta'] ; \dots ; [(d^{(r)}) : \delta^{(r)}] ; z_r 2^{h_r+k_r} \end{aligned} \quad \dots (2.5)$$

where

$$\operatorname{Re}(\eta) > -1, \operatorname{Re}(\sigma) > -1, h_i > 0, k_i > 0,$$

$$|\arg(z_i)| < \frac{1}{2} T_i \pi, \text{ and } \operatorname{Re}\left(\eta + \sum_{i=1}^r h_i d_j^{(i)} / \delta_j^{(i)}\right) > -1.$$

$$\operatorname{Re}\left(\sigma + \sum_{i=1}^r k_i d_j^{(i)}\right) > 1, \quad i = 1, \dots, r$$

$$\int_{-1}^1 (1-x)^{\alpha} (1+x)^{\sigma} \frac{1}{\rho} (1-t+\rho)^{-\alpha} (1+t+\rho)^{-\beta} H \left(\begin{matrix} z_1 (1-x)^{h_1} (1+x)^{k_1} \\ z_r (1-x)^{h_r} (1+x)^{k_r} \end{matrix} \right) dx$$

$$= 2^{-\alpha-\beta+\eta+\sigma+1} \sum_{R=0}^{\infty} \sum_{m=0}^R t^R \frac{(-R)_m (\alpha+\beta+R+1)_m}{(\alpha+1)_m m!}$$

$$\begin{aligned} & 0, \varepsilon+2 : (M', N') ; \dots ; (M^{(r)}, N^{(r)}) \\ \cdot H & A+2, C+1 : (B', D') ; \dots, (B^{(r)}, D^{(r)}) \end{aligned}$$

$$\begin{aligned}
 & \left([(a) : \theta', \dots, \theta^{(r)}], (-\sigma : k_1, \dots, k_r), (-m-\eta : h_1, \dots, h_r) : \right. \\
 & \left. [(c) : C', \dots, C^{(r)}], (-1-m-\eta-\sigma : h_1 + k_1, \dots, h_r + k_r) : \right. \\
 & [(b') : \phi'] ; \dots ; [(b^{(r)}) : \phi^{(r)}] ; z_1 2^{h_1+k_1} \\
 & [(d') : \delta'] ; \dots ; [(d^{(r)}) : \delta^{(r)}] ; z_r 2^{h_r+k_r} \left. \right) \dots (2.6)
 \end{aligned}$$

where

$$Re(\eta) > -1, Re(\sigma) > -1, h_i > 0, k_i > 0,$$

$$|\arg(z_i)| < \frac{1}{2} T_i \pi, > 0 \text{ and}$$

$$Re\left(\eta + \sum_{i=1}^r h_i d_j^{(i)} / \delta_j^{(i)}\right) > -1,$$

$$Re\left(\sigma + \sum_{i=1}^r k_i d_j^{(i)} / \delta_j^{(i)}\right) > -1, i = 1, \dots, r$$

Proof : To prove (2.1), multiply both the sides of (1.7) by

$$(1+x)^\eta (1-x)^\alpha H\left(\begin{matrix} z_1 (1+x)^{h_1} \\ \vdots \\ z_r (1+x)^{h_r} \end{matrix}\right) \text{ and integrating both sides with}$$

respect to x between the limits -1 to 1 , we get

$$\int_{-1}^1 (1+x)^\eta (1-x)^\alpha P_k^{(\alpha, \beta)}(t+\rho) P_k^{(\alpha, \beta)}(t-\rho) H\left(\begin{matrix} z_1 (1+x)^{h_1} \\ \vdots \\ z_r (1+x)^{h_r} \end{matrix}\right) dx$$

$$= \int_{-1}^1 \frac{(-1)^k (1+\alpha)_k (1+\beta)_k}{(k!)^2} \sum_{R=0}^k \frac{(-k)_R}{(1+\alpha)_R} \frac{(1+\alpha+\beta+k)_R}{(1+\beta)_R}$$

$$\cdot P_k^{(\alpha, \beta)}(x) t^R (1+x)^\eta (1-x)^\alpha H\left(\begin{matrix} z_1 (1+x)^{h_1} \\ \vdots \\ z_r (1+x)^{h_r} \end{matrix}\right) dx \dots (2.7)$$

Now, on interchanging the order of integration and summation in the right-hand side of (2.7), which is permissible as the series involved is finite and integrals involved are absolutely convergent and on evaluating the x -integral thus obtained with the help of (1.10), we get the desired result under the conditions mentioned therein.

The proofs of the formulas (2.2) to (2.6) can be developed by proceeding on similar lines with the help of the result (1.8) and (1.10), (1.9) and (1.10), (1.9) and (1.11), (1.8) and (1.11), (1.9) and (1.11), respectively.

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SOME EXPANSION FORMULAE FOR THE MULTIVARIABLE H - FUNCTION

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ABSTRACT

The object of the present paper is to derive some expansion formulae for the multivariable H -function involving exponential functions, Legendre functions and Bessel functions .

1. INTRODUCTION

Srivastava and Panda [10] have defined the multivariable H -function as (see also [9] and [11])

$$H^{0, \epsilon : (u', v') ; \dots ; (u^{(r)}, v^{(r)})}$$

$$A, C : (B', D') ; \dots ; (B^{(r)}, D^{(r)})$$

$$\left[\begin{array}{l} [(a) : \theta', \dots, \theta^{(r)}] : [(b') : \phi', \dots, [(b^{(r)}) : \phi^{(r)}] ; \\ [(c) : \Psi', \dots, \Psi^{(r)}] : [(d') : \delta', \dots, [(d^{(r)}) : \delta^{(r)}] ; \end{array} z_1, \dots, z_r \right]$$

$$= \frac{1}{(2\pi i)^r} \int_{L_1} \dots \int_{L_r} U_1(s_1) \dots U_r(s_r) V(s_1, \dots, s_r)$$

$$z_1^{s_1} \dots z_r^{s_r} ds_1 \dots ds_r \dots (1.1)$$

$$U_i(s_i) = \frac{\prod_{j=1}^{u^{(i)}} \Gamma(d_j^{(i)} - \delta_j^{(i)} s_i) \prod_{j=1}^{v^{(i)}} \Gamma(1 - b_j^{(i)} + \phi_j^{(i)} s_i)}{\prod_{j=1+u^{(i)}}^{D^{(i)}} \Gamma(1 - d_j^{(i)} + \delta_j^{(i)} s_i) \prod_{j=v^{(i)}+1}^{B^{(i)}} \Gamma(b_j^{(i)} - \phi_j^{(i)} s_i)} \quad i = 1, \dots, r \quad \dots (1.2)$$

and

$$V(s_1, \dots, s_r) = \frac{\prod_{j=1}^{\epsilon} \Gamma(1 - a_j + \sum_{i=1}^r \theta_j^{(i)} s_i)}{\prod_{j=\epsilon+1}^A \Gamma(a_j - \sum_{i=1}^r \theta_j^{(i)} s_i) \prod_{j=1}^C \Gamma(1 - c_j + \sum_{i=1}^r \psi_j^{(i)} s_i)} \quad \dots (1.3)$$

and, for convergence,

$$| \arg(z_i) | < \frac{1}{2} T_i \pi, \quad i = 1, \dots, r \quad \dots (1.4)$$

where,

$$T_i = \frac{A}{-\sum_{j=\epsilon+1} \theta_j^{(i)}} + \sum_{j=1}^{v^{(i)}} \theta_j^{(i)} - \sum_{j=v^{(i)}+1}^{B^{(i)}} \theta_j^{(i)} - \sum_{j=1}^C \psi_j^{(i)} + \sum_{j=1}^{u^{(i)}} \delta_j^{(i)} - \sum_{j=u^{(i)}+1}^{D^{(i)}} \delta_j^{(i)} > 0, \quad i = 1, \dots, r \quad \dots (1.5)$$

2. PRELIMINARY RESULTS

The following known results [7, p. 340 (95) ; 4, p. 316 ; 8, p. 78 ; 3, p. 62 ; 12, p. 324 ; 6, p. 291 (6)] will be required in the proof of the integrals and expansions:

$$\int_{-\pi/2}^{\pi/2} (\cos \theta)^{m+n-2} e^{i(m-n)\theta} d\theta = \frac{\pi \Gamma(m+n-1)}{2^{m+n-2} \Gamma(m) \Gamma(n)}, \quad \text{Re}(m+n) > 1$$

... (2.1)

$$\int_{-1}^1 (1-x^2)^{\alpha-1} P_{\beta}^{\delta}(x) dx = \frac{2^{\delta} \pi \Gamma(\alpha+\delta/2) \Gamma(\alpha-\delta/2)}{\Gamma(\alpha+\frac{\beta}{2}+1) \Gamma(\alpha-\frac{\beta}{2}) \Gamma(\frac{2-\delta+\beta}{2}) \Gamma(\frac{1-\beta-\delta}{2})}$$

... (2.2)

$$\text{Re}(2\alpha + \delta) > 0, \text{Re}(2\alpha - \delta) > 0.$$

$$\int_0^{\infty} e^{ix} x^{\alpha} J_{\beta}(x) dx = \frac{e^{1/2(i(\alpha+\beta+1)\pi)} \Gamma(\alpha+\beta+1) \Gamma(-\alpha-\frac{1}{2})}{2^{\alpha+1} \Gamma(\frac{1}{2}) \Gamma(\beta-\alpha)}$$

... (2.3)

$$\text{Re}(\alpha+\beta) > -1, \text{Re}(\alpha) < -\frac{1}{2}.$$

$$\int_0^{\infty} t^{s-1} J_{\beta}(\alpha t) J_{\delta}(\alpha t) dt = \frac{2^{s-1} \alpha^{-s} B(1-s, \frac{\beta+\delta+s}{2})}{\Gamma(\frac{\delta-\beta-s}{2}+1) \Gamma(\frac{\beta-\delta-s}{2}+1)}$$

... (2.4)

$$-\text{Re}(\beta+\delta) < \text{Re}(s) < 1, \alpha > 0.$$

$$\int_a^b e^{\frac{2m\pi ix}{a-b}} e^{\frac{2n\pi ix}{a-b}} dx = 0, \text{ if } m \neq -n$$

... (2.5)

$$= b-a, \text{ if } m = -n$$

$$\int_{-1}^1 P_n^m(x) P_r^m(x) dx = 0, \text{ if } r \neq n$$

$$= \frac{2}{2n+1} \cdot \frac{(n+m)!}{(n-m)!} \quad \text{if } r = n$$

... (2.6)

$$\int_0^{\infty} t^{-1} J_{\alpha+2n+1}(t) J_{\alpha+2m+1}(t) dt = 0, \quad \text{if } m \neq n$$

$$= (4n+2\alpha+2)^{-1}, \text{ if } m = n \quad \dots (2.7)$$

$$\operatorname{Re}(\alpha) + m + n > -1.$$

3. MAIN INTEGRALS

$$\int_{-\pi/2}^{\pi/2} (\cos \theta)^{m+n-2} e^{i(m-n)\theta} H_{0, \varepsilon : (u', v') ; \dots ; (u^{(r)}, v^{(r)})} A, C : (B', D') ; \dots ; (B^{(r)}, D^{(r)})$$

$$\left[(a) : \theta', \dots, \theta^{(r)} \right] : [(b') : \phi'] ; \dots ; [(b^{(r)}) : \phi^{(r)}] ;$$

$$\left[(c) : \Psi', \dots, \Psi^{(r)} \right] : [(d') : \delta'] ; \dots ; [(d^{(r)}) : \delta^{(r)}] ;$$

$$(e^{i\theta} \cos \theta)^{h_1/2} z_1, z_2, \dots, z_r \Big] d\theta$$

$$= \frac{2}{2^{m+n} \Gamma(n)} \sum_{i,-i} \frac{1}{i} H_{0, \varepsilon : (u'+1, v'+1) ; (u'', v'')} A, C : (B'+2, D'+2) ; (B'', D'') ;$$

$$\dots ; (u^{(r)}, v^{(r)}) \left[(a) : \theta', \dots, \theta^{(r)} \right] : [(b') : \phi'], (1, h_1),$$

$$\dots ; (B^{(r)}, D^{(r)}) \left[(c) : \Psi', \dots, \Psi^{(r)} \right] : [(d') : \delta'], (1, h_1),$$

$$(2-m-n, \frac{h_2}{2}) ; [(b'') : \phi''] ; \dots ; [(b^{(r)}) : \delta^{(r)}] ;$$

$$(1-m, \frac{h_1}{2}) ; [(d'') : \phi''] ; \dots ; [(d^{(r)}) : \delta^{(r)}] ;$$

$$\frac{e^{i\pi h_1}}{2h_1/2} z_1, z_2, \dots, z_r \Big] \quad \dots (3.1)$$

where h_1 is a positive number and $\operatorname{Re}(m+n) > 1$ and conditions of (1.1) are also satisfied. The symbol $\sum_{i,-i}$ means that in the expression

following it, i is to be replaced by $-i$ and the two expressions so obtained must be added.

$$\begin{aligned}
 & \int_{-1}^1 (1-x^2)^{\alpha-1} P_{\beta}^{\delta}(x) H_{0, \varepsilon : (u', v') ; \dots ; (u^{(r)}, v^{(r)})} \\
 & \quad A, C : (B', D') ; \dots ; (B^{(r)}, D^{(r)}) \\
 & \left[\begin{aligned} & [(a) : \theta', \dots, \theta^{(r)}] : [(b') : \phi'] ; \dots ; [(b^{(r)}) : \phi^{(r)}] ; \\ & [(c) : \Psi', \dots, \Psi^{(r)}] : [(d') : \delta'] ; \dots ; [(d^{(r)}) : \delta^{(r)}] ; \end{aligned} \right. \\
 & \quad \left. z_1 (1-x^2)^{h_1}, \dots, z_r (1-x^2)^{h_r} \right] dx \\
 & = \frac{2\delta\pi}{\Gamma(\frac{2-\delta+\beta}{2}) \Gamma(\frac{1-\delta-\beta}{2})} H_{0, \varepsilon+2 : (u', v') ; \dots ; (u^{(r)}, v^{(r)})} \\
 & \quad A+2, C+2 : (B', D') ; \dots ; (B^{(r)}, D^{(r)}) \\
 & \left[\begin{aligned} & [(a) : \theta', \dots, \theta^{(r)}], (1-\alpha-\frac{\delta}{2} ; h_1, \dots, h_r), (1-\alpha+\frac{\delta}{2} ; h_1, \dots, h_r) : \\ & [(c) : \Psi', \dots, \Psi^{(r)}], (-\alpha-\frac{\beta}{2} ; h_1, \dots, h_r), (1-\alpha+\frac{\beta}{2} ; h_1, \dots, h_r) : \end{aligned} \right. \\
 & \quad \left. \begin{aligned} & [(b') : \phi'] ; \dots ; [(b^{(r)}) : \phi^{(r)}] ; \\ & [(d') : \delta'] ; \dots, [(d^{(r)}) : \delta^{(r)}] ; \end{aligned} \right]_{z_1, \dots, z_r} \dots (3.2)
 \end{aligned}$$

where

$$|\arg(z_i)| < \frac{1}{2} T_i \pi, T_i > 0, \text{ and}$$

$$2 \operatorname{Re}(\alpha + \sum_{i=1}^r h_i \frac{d_j^{(i)}}{\delta_j^{(i)}}) > |\operatorname{Re}(\delta)|, j = 1, \dots, u^{(i)}.$$

$$\int_0^\infty e^{iz} x^\alpha J_\beta(x) H_{A,C:(B',D');\dots;(B^{(r)},D^{(r)})}^{0,\varepsilon:(u',v');\dots;(u^{(r)},v^{(r)})} \left[\begin{matrix} [(a):\theta',\dots,\theta^{(r)}]; \\ [(c):\Psi',\dots,\Psi^{(r)}]; \end{matrix} \right.$$

$$\left. \begin{matrix} [(b'):\phi']; \dots; [(b^{(r)}):\phi^{(r)}]; \\ [(d'):\delta']; \dots; [(d^{(r)}):\delta^{(r)}]; \end{matrix} \right] x^{h_1} z_1, z_2, \dots, z_r dx$$

$$\frac{e^{\frac{1}{2}i(\alpha+\beta+1)\pi}}{\Gamma(\frac{1}{2})2^{\alpha+1}} H_{A,C:(B'+2,D'+1);(B'',D'');\dots;(B^{(r)},D^{(r)})}^{0,\varepsilon:(u'+1,v'+1);(u'',v'');\dots;(u^{(r)},v^{(r)})}$$

$$\left[\begin{matrix} [(a):\theta',\dots,\theta^{(r)}]:[(b'):\phi'],(-\alpha-\beta,h_1),(\beta-\alpha,h_1);[(b''):\phi'']; \\ [(c):\Psi',\dots,\Psi^{(r)}]:[(d'):\delta'],(-\alpha-\frac{1}{2},h_1) \end{matrix} \right]; [(d''):\delta''] | 1$$

$$\left. \begin{matrix} \dots; [(b^{(r)}):\phi^{(r)}]; \\ \dots; [(d^{(r)}):\delta^{(r)}]; \end{matrix} \right] \left(\frac{e^{\frac{1}{2}i\pi}}{2} \right) h_1 z_1, z_2, \dots, z_r$$

... (3.3)

where $| \arg(z_i) | < \frac{1}{2} T_i \pi$, $T_i > 0$, and

$$\operatorname{Re}(\alpha + \beta + h_1 \frac{d_j^{(i)}}{\delta_j^{(i)}}) > -1, \operatorname{Re}(\alpha) < \frac{1}{2}, j=1,2,\dots,u^{(i)}.$$

$$\int_0^\infty x^{-k} J_\beta(\alpha x) J_\beta(\alpha x) H_{A,C:(B',D');\dots;(B^{(r)},D^{(r)})}^{0,\varepsilon:(u',v');\dots;(u^{(r)},v^{(r)})}$$

$$\left[\begin{matrix} [(a):\theta',\dots,\theta^{(r)}]:[(b'):\phi']; \dots; [(b^{(r)}):\phi^{(r)}]; \\ [(c):\Psi',\dots,\Psi^{(r)}]:[(d'):\delta']; \dots; [(d^{(r)}):\delta^{(r)}]; \end{matrix} \right]$$

$$z_1 x^{2h_1}, z_2, \dots, z_r dx$$

$$= \frac{1}{2\sqrt{\pi}} a^{k-1} H_{0,\varepsilon : (u'+2, v'+1); (u'', v''); A, C : (B'+4, D'+2); (B'', D'') ;$$

$$\dots; (u^{(r)}, v^{(r)}) \left[[(a) : \theta', \dots, \theta^{(r)}] : [(b') : \phi'] , \right. \\ \left. \dots; (B^{(r)}, D^{(r)}) \left[[(c) : \Psi', \dots, \Psi^{(r)}] : [(d') : \delta'] , \right. \right.$$

$$\left(\frac{\beta}{2} - \frac{\delta}{2} + \frac{k}{2} + \frac{1}{2}, h_1 \right),$$

$$\left(\frac{k}{2}, h_1 \right), \left(\frac{k}{2} + \frac{1}{2}, h_1 \right)$$

$$\left(\frac{1}{2} + \frac{k}{2} - \frac{\beta}{2} - \frac{\delta}{2}, h_1 \right), \left(\frac{1}{2} + \frac{\delta}{2} + \frac{\beta}{2} + \frac{k}{2}, h_1 \right),$$

$$\left(\frac{\delta}{2} - \frac{\beta}{2} + \frac{k}{2} + \frac{1}{2}, h_1 \right); [(b'') : \phi''] ; \dots; [(b^{(r)}) : \phi^{(r)}] ; \\ \dots; [(d'') : \phi''] ; \dots; [(d^{(r)}) : \delta^{(r)}] ;$$

$$\left[z_1 a^{-2h_1}, z_2, \dots, z_r \right] \dots (3.4)$$

where $|\arg(z_i)| < \frac{1}{2} T_i \pi$, $T_i > 0$, and

$$\operatorname{Re} \left(\beta + \delta - k + 2h_1 \frac{d_j^{(i)}}{\delta_j^{(i)}} \right) > -1,$$

$$\operatorname{Re} \left(2h_1 \frac{d_j^{(i)}}{\delta_j^{(i)}} - k \right) < 2h_1, j=1, \dots, u^{(i)}$$

Proof: Equation (3.1) can be proved on using the contour integral (1.1), interchanging the order of integration, which is justified under

the conditions stated with the result, evaluating the snner integral with the help of (2.1) and using a relation $\Gamma(a) \Gamma(1-a) = \frac{\pi}{\sin \pi a}$

$$= \frac{2\pi i}{e^{i\pi\alpha} - e^{-i\pi\alpha}}, \text{ and applying the definition (1.1).}$$

The proof of the formula (3.2) to (3.4) can be developed by proceeding on similar lines with the help of the result (2.2) to (2.4) respectively in place of result (2.1) and a relation $\Gamma(a) \Gamma(1-a) = \frac{\pi}{\sin \pi a}$.

4. EXPANSION FORMULAE

$$\begin{aligned} & (\cos \theta)^{2n-2} H^{0, \epsilon : (u', v') ; \dots ; (u^{(r)}, v^{(r)})} \left[[(a) : \theta' , \dots , \theta^{(r)}] : \right. \\ & \quad \left. A, C : (B', D') ; \dots ; (B^{(r)}, D^{(r)}) \right] [(c) : \Psi' , \dots , \Psi^{(r)}] : \\ & \quad [(b') : \phi'] ; \dots ; [(b^{(r)}) : \phi^{(r)}] ; \\ & \quad [(d') : \delta'] ; \dots ; [(d^{(r)}) : \delta^{(r)}] ; \left(e^{i\theta} \cos \theta \right)^{h_1/2} z_1, z_2, \dots, z_r \Big] \\ & = 2 \left\{ \sum_{R=-\infty}^{\infty} \frac{1}{2^{2R} \Gamma(R)} \sum_{i, -i} \frac{1}{i} \right. \\ & \quad H^{0, \epsilon : (u'+1, v'+2) ; (u'', v'') ; \dots ; (u^{(r)}, v^{(r)})} \\ & \quad \left. A, C : (B'+2, D'+2) ; (B'', D'') ; \dots ; (B^{(r)}, D^{(r)}) \right. \\ & \quad \left[[(a) : \theta' , \dots , \theta^{(r)}] : [(b') : \phi'], (1, h_1), (2-2R, \frac{h_1}{2}) ; [(b'') : \phi''] ; \right. \\ & \quad \left. [(c) : \Psi' , \dots , \Psi^{(r)}] : [(d') : \delta'], (1, h_1), (1-R, \frac{h_1}{2}) ; [(d'') : \delta''] ; \right. \\ & \quad \left. \dots ; [(b^{(r)}) : \phi^{(r)}] ; \frac{e^{i\pi h_1}}{2^{h_1/2}} z_1, z_2, \dots, z_r \right\} e^{-2i(R-n)\theta} \\ & \quad \dots (4.1) \end{aligned}$$

where h_1 is a positive number, $-\pi/2 < \theta < \pi/2$, and $\operatorname{Re}(n) > 1/2$.

$$(1-x^2)^{\alpha-1} H \begin{matrix} 0, \varepsilon : (u', v') ; \dots ; (u^{(r)}, v^{(r)}) \\ A, C : (B', D') ; \dots ; (B^{(r)}, D^{(r)}) \end{matrix}$$

$$\left[\begin{matrix} [(a) : \theta', \dots, \theta^{(r)}] : [(b') : \phi'] ; \dots ; [(b^{(r)}) : \phi^{(r)}] ; \\ [(c) : \Psi', \dots, \Psi^{(r)}] : [(d') : \delta'] ; \dots ; [(d^{(r)}) : \delta^{(r)}] ; \\ z_1 (1-x^2)^{h_1}, \dots, z_r (1-x^2)^{h_r} \end{matrix} \right]$$

$$= 2^{\delta-1} \pi \sum_{R=0}^{\infty} \frac{(2R+1)(R-\delta)!}{(R+\delta)! \Gamma\left(\frac{2-\delta+R}{2}\right) \Gamma\left(\frac{1-\delta-R}{2}\right)}$$

$$H \begin{matrix} 0, \varepsilon+2 : (u', v') ; \dots ; (u^{(r)}, v^{(r)}) \\ A+2, C+2 : (B', D') ; \dots ; (B^{(r)}, D^{(r)}) \end{matrix}$$

$$\left[\begin{matrix} [(a) : \theta', \dots, \theta^{(r)}], (1-\alpha-\frac{\delta}{2}; h_1, \dots, h_r), (-\alpha+\frac{\delta}{2}; h_1, \dots, h_r); \\ [(c) : \Psi', \dots, \Psi^{(r)}], (-\alpha-\frac{R}{2}; h_1, \dots, h_r), (1-\alpha+\frac{R}{2}; h_1, \dots, h_r); \end{matrix} \right]$$

$$\left[\begin{matrix} [(b') : \phi'] ; \dots ; [(b^{(r)}) : \phi^{(r)}] ; \\ [(d') : \delta'] ; \dots, [(d^{(r)}) : \delta^{(r)}] ; \end{matrix} z_1, \dots, z_r \right] P_R^{\delta}(x) \dots (4.2)$$

where

$$|\arg(z_i)| < \frac{1}{2} T_i \pi, T_i > 0, \text{ and}$$

$$\operatorname{Re}(\alpha + \sum_{i=1}^r h_i \frac{d_j^{(i)}}{\delta_j^{(i)}}) > |\operatorname{Re}(\delta)|, j = 1, \dots, u^{(1)};$$

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$$\begin{aligned}
& x^\alpha H \begin{matrix} 0, \varepsilon : (u', v') ; \dots ; (u^{(r)}, v^{(r)}) \\ A, C : (B', D') ; \dots ; (B^{(r)}, D^{(r)}) \end{matrix} \left[\begin{matrix} [(a) : \theta', \dots, \theta^{(r)}] \\ [(c) : \Psi', \dots, \Psi^{(r)}] \end{matrix} \right] \\
& \left[\begin{matrix} [(b') : \phi'] ; \dots ; [(b^{(r)}) : \phi^{(r)}] ; \\ [(d') : \delta'] ; \dots ; [(d^{(r)}) : \delta^{(r)}] ; \end{matrix} \right]_{x^{h_1} z_1, z_2, \dots, z_r} \\
& = \frac{1}{\Gamma(\frac{1}{2}) 2^{\alpha-1}} \sum_{R=0}^{\infty} e^{\frac{1}{2} i(a + \gamma_1) \pi} \\
& \eta H \begin{matrix} 0, \varepsilon : (u' + 1, v' + 1) ; (u'', v'') ; \dots ; (u^{(r)}, v^{(r)}) \\ A, C : (B' + 2, D' + 1) ; (B'', D'') ; \dots ; (B^{(r)}, D^{(r)}) \end{matrix} \\
& \left[\begin{matrix} [(a) : \theta', \dots, \theta^{(r)}] : [(b') : \phi'] , (-\alpha + 1 - \gamma_1 h_1) , \\ [(c) : \Psi', \dots, \Psi^{(r)}] : [(d') : \delta'] , (-\alpha + \frac{1}{2}, h_1) \end{matrix} \right] \\
& (\gamma_1 - \alpha + 1 ; h_1) ; [(b'') : \phi''] ; \dots ; [(b^{(r)}) : \phi^{(r)}] ; \\
& \quad ; [(d'') : \delta''] ; \dots ; [(d^{(r)}) : \delta^{(r)}] ; \\
& \left(\frac{e^{\frac{1}{2} i \pi}}{2} \right)_{z_1, z_2, \dots, z_r}^{h_1} \left[J_\eta(x) \right] \dots (4.3)
\end{aligned}$$

where $\eta = n + 2R + 1$,

$| \arg(z_i) | < \frac{1}{2} T_i \pi, T_i > 0$, and

$$Re(\alpha + h_1 \frac{d_j^{(i)}}{\delta_j^{(i)}}) > 0, Re \alpha < \frac{1}{2}, j=1, \dots, u^{(i)}.$$

$$\begin{aligned}
& x^{-k} J_\beta(\alpha x) H \begin{matrix} 0, \varepsilon : (u', v') ; \dots ; (u^{(r)}, v^{(r)}) \\ A, C : (B', D') ; \dots ; (B^{(r)}, D^{(r)}) \end{matrix} \left[\begin{matrix} [(a) : \theta', \dots, \theta^{(r)}] : \\ [(c) : \Psi', \dots, \Psi^{(r)}] ; \end{matrix} \right] \\
& \left[\begin{matrix} [(b') : \phi'] ; \dots ; [(b^{(r)}) : \phi^{(r)}] ; \\ [(d') : \delta'] ; \dots ; [(d^{(r)}) : \delta^{(r)}] ; \end{matrix} \right]_{x^{2h_1} z_1, z_2, \dots, z_r}
\end{aligned}$$

$$= \frac{\alpha^k}{\sqrt{\pi}} \sum_{R=0}^{\infty} \eta H_{0, \varepsilon : (u'+2, v'+1); (u'', v''); \dots; (u^{(r)}, v^{(r)})} \\ A, C : (B'+4, D'+2); (B'', D''); \dots; (B^{(r)}, D^{(r)})$$

$$\left[\begin{array}{l} [(a) : \theta', \dots, \theta^{(r)}] : [(b') : \phi'], (\frac{\beta}{2} - \frac{\eta}{2} + \frac{k}{2} + 1, h_1), \\ [(c) : \Psi', \dots, \Psi^{(r)}] : [(d') : \delta'], (\frac{k}{2} + \frac{1}{2}, h_1) \end{array} \right],$$

$$(1 + \frac{k}{2} - \frac{\beta}{2} - \frac{\eta}{2}, h_1), (1 + \frac{\eta}{2} + \frac{\beta}{2} + \frac{k}{2}, h_1), (\frac{\eta}{2} - \frac{\beta}{2} + \frac{k}{2} + 1, h_1); \\ (\frac{k}{2} + 1, h_1);$$

$$[(b'') : \phi''] ; \dots ; [(b^{(r)}) : \phi^{(r)}] ; \\ [(d'') : \delta''] ; \dots ; [(d^{(r)}) : \delta^{(r)}] ; \left. z_1^{\alpha-2h_1}, z_2, \dots, z_r \right] J_{\eta}(x), \dots (4.4)$$

where $\eta = u + 2R + 1, |arg(z_i)| < \frac{1}{2} T_i \pi, T_i > 0$, and

$$Re(\beta - k + 2h_1 \frac{d_j^{(i)}}{\delta_j^{(i)}}) > 0,$$

$$Re(2h_1 \frac{b_j^{(i)}}{\phi_j^{(i)}} - k) < 2h_1 + 1, j=1, \dots, u^{(i)}.$$

Proof: To prove (4.1), let

$$f(\theta) = (\cos \theta)^{2n-2} H_{0, \varepsilon : (u', v'); \dots; (u^{(r)}, v^{(r)})} \left[\begin{array}{l} [(a) : \theta', \dots, \theta^{(r)}] : \\ A, C : (B', D'); \dots; (B^{(r)}, D^{(r)}) \end{array} \right] \\ \left[\begin{array}{l} [(b') : \phi']; \dots; [(b^{(r)}) : \phi^{(r)}]; \\ [(d') : \delta']; \dots; [(d^{(r)}) : \delta^{(r)}]; \end{array} \right] (e^{i\theta} \cos \theta)^{h_1/2} z_1, z_2, \dots, z_r$$

$$= \sum_{R=-\infty}^{\infty} C_R e^{-2i(R-n)\theta}, \quad \dots (4.5)$$

$$-\frac{\pi}{2} < \theta < \frac{\pi}{2}, \quad \operatorname{Re}(n) > 0.$$

Equation (4.5) is valid, since $f(\theta)$ is continuous and of bounded variation in the open intervals $(-\frac{\pi}{2} \text{ to } \frac{\pi}{2})$.

Now integrating (4.5) with respect to θ from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$, we get

$$\int_{-\pi/2}^{\pi/2} f(\theta) d\theta = \sum_{R=-\infty}^{\infty} C_R \int_{-\pi/2}^{\pi/2} e^{-2i(R-n)\theta} d\theta$$

Now using (3.1) and the orthogonality property of exponential functions (2.5), we obtain

$$C_n = \frac{2}{2^{2n} \Gamma(n)} \sum_{i, -i} \frac{1}{i}$$

$$H \quad 0, \varepsilon : (u' + 1, v' + 2); (u'', v''); \dots; (u^{(r)}, v^{(r)})$$

$$A, C : B' + 2, D' + 2; (B'', D''); \dots; (B^{(r)}, D^{(r)})$$

$$\left[\begin{array}{l} [(a) : \theta', \dots, \theta^{(r)}] : [(b') : \phi'], (1, h_1), (2-2n, h_1/2); \\ [(c) : \Psi', \dots, \Psi^{(r)}] : [(d') : \delta'], (1, h_1), (1-n, h_1/2); \end{array} \right.$$

$$[(b'') : \phi'']; \dots; [(b^{(r)}) : \phi^{(r)}];$$

$$[(d'') : \delta'']; \dots; [(d^{(r)}) : \delta^{(r)}];$$

$$\frac{e^{i\pi h_1}}{2^{h_1/2}} \left. \begin{array}{l} z_1, z_2, \dots, z_r \end{array} \right] \quad \dots (4.6)$$

With the help of equations (4.5) and (4.6), the formula (4.1) follows. The proofs of the formulas (4.2) to (4.4) can be developed by proceeding on similar lines with the help of (3.2) and (2.6), (3.3) and (2.7), (3.4) and (2.7), respectively.

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CERTAIN RESULTS INVOLVING A CLASS OF GENERALIZED LAURICELLA FUNCTIONS

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1. INTRODUCTION

Certain addition and multiplication theorems of Lauricella functions have been discussed by H. Exton [1]. Our aim, in the present note, is to establish some results concerning the generalized Lauricella functions by an application of a fractional differential operator.

2. Notations and Definitions

We consider the following special class of the generalized Lauricella functions due to Srivastava and Daoust ([2, p. 454] ; see also [1, p. 107]) :

$$F \begin{matrix} 1 : 2 ; \dots ; 2 \\ 0 : 2 ; \dots ; 2 \end{matrix} \left(\begin{matrix} a : b_1, \lambda_1 ; \dots ; b_n, \lambda_n ; \\ - : c_1, \mu_1 ; \dots ; c_n, \mu_n ; \end{matrix} \begin{matrix} x_1, \dots, x_n \end{matrix} \right)$$

$$= \sum_{m_1, \dots, m_n=0}^{\infty} \frac{(a)_{m_1+\dots+m_n} (b_1)_{m_1} (\lambda_1)_{m_1} \dots (b_n)_{m_n} (\lambda_n)_{m_n} x_1^{m_1} \dots x_n^{m_n}}{(c_1)_{m_1} (\mu_1)_{m_1} \dots (c_n)_{m_n} (\mu_n)_{m_n} (1)_{m_1} \dots (1)_{m_n}}$$

Also, the fractional differential operator D_x^δ is defined by (see, e. g., [3, p. 289])

$$D_a^\delta x^{a-1} = \frac{\Gamma(a)}{\Gamma(a-\delta)} x^{a-\delta-1},$$

where $\delta \neq a$ and $a > 0$.

3. Main Results

In this section, we shall establish the following results :

$$(3.1) \quad \sum_{m_1, \dots, m_n=0}^{\infty} \frac{(a)_{m_1+\dots+m_n} (b_1)_{m_1} (\lambda_1)_{m_1} \dots (b_n)_{m_n}}{(c_1)_{m_1} (\lambda_1-\delta_1)_{m_1} \dots (c_n)_{m_n} (\lambda_n-\delta_n)_{m_n}}.$$

$$\frac{(\lambda_n)_{m_n} x_1^{m_1} \dots x_n^{m_n}}{(1)_{m_1} \dots (1)_{m_n}}$$

$$\cdot F \quad \begin{matrix} 1:2;\dots;2 \\ 0:2;\dots;2 \end{matrix} \left(\begin{matrix} a+m_1+\dots+m_n : b_1+m_1, \mu_1; \dots; \\ - : c_1+m_1, \mu_1-\nu_1; \dots; \\ b_n+m_n, \mu_n; \\ c_n+m_n, \mu_n-\nu_n; \end{matrix} \right. \left. \begin{matrix} y_1, \dots, y_n \end{matrix} \right)$$

$$= \sum_{p_1, \dots, p_n=0}^{\infty} \frac{(a)_{p_1+\dots+p_n} (b_1)_{p_1} (\mu_1)_{p_1} \dots (b_n)_{p_n}}{(c_1)_{p_1} (\mu_1-\nu_1)_{p_1} \dots (c_n)_{p_n} (\mu_n-\nu_n)_{p_n}}.$$

$$\frac{(\mu_n)_{p_n} y_1^{p_1} \dots y_n^{p_n}}{(1)_{p_1} \dots (1)_{p_n}}$$

$${}_1F_0 \left(\begin{matrix} 1:2; \dots; 2 \\ 0:2; \dots; 2 \end{matrix} \left(\begin{matrix} a+p_1+\dots+p_n : b_1+p_1, \lambda_1; \dots; \\ - : c_1+p_1, \lambda_1-\delta_1; \dots; \\ b_n+p_n, \lambda_n; \\ c_n+p_n, \lambda_n-\delta_n; \end{matrix} x_1, \dots, x_n \right) \right).$$

$$(3,2) {}_1F_0 \left(\begin{matrix} 1:3; \dots; 3 \\ 0:3; \dots; 3 \end{matrix} \left(\begin{matrix} a : \lambda_1, b_1, \mu_1; \dots; \lambda_n, b_n, \mu_n; \\ - : \lambda_1-\delta_1, c_1, \mu_1-\nu_1; \dots; \lambda_n-\delta_n, c_n, \mu_n-\nu_n; \\ x_1 y_1, \dots, x_n y_n \end{matrix} \right) \right)$$

$$= \sum_{p_1, \dots, p_n=0}^{\infty} \frac{(a)_{p_1+\dots+p_n} (b_1)_{p_1} (\lambda_1)_{p_1} \dots (b_n)_{p_n} (\lambda_n)_{p_n}}{(c_1)_{p_1} (\lambda_1-\delta_1)_{p_1} \dots (c_n)_{p_n} (\lambda_n-\delta_n)_{p_n}} \frac{(-x_1)^{p_1} \dots (-x_n)^{p_n}}{(1)_{p_1} \dots (1)_{p_n}}$$

$${}_1F_0 \left(\begin{matrix} 1:2; \dots; 2 \\ 0:2; \dots; 2 \end{matrix} \left(\begin{matrix} a+p_1+\dots+p_n : b_1+p_1, \lambda_1+p_1; \dots; \\ - : c_1+p_1, \lambda_1-\delta_1+p_1; \dots; \\ b_n+p_n, \lambda_n+p_n; \\ c_n+p_n, \lambda_n-\delta_n+p_n; \end{matrix} x_1, \dots, x_n \right) \right)$$

$$\prod_{i=1}^n {}_2F_1 \left(\begin{matrix} \mu_i, -p_i; \\ \mu_i-\nu_i; \end{matrix} y_i \right)$$

4. Proofs

Proof of (3.1): The addition theorem for Lauricella function $F_A^{(n)}$, given by H. Exton [1], is

$$(4.1) \quad F_A^{(n)}(a, b_1, \dots, b_n; c_1, \dots, c_n; x_1+y_1, \dots, x_n+y_n)$$

$$= \sum_{p_1, \dots, p_n=0}^{\infty} \frac{(a)_{p_1+\dots+p_n} (b_1)_{p_1} \dots (b_n)_{p_n} y_1^{p_1} \dots y_n^{p_n}}{(c_1)_{p_1} \dots (c_n)_{p_n} (1)_{p_1} \dots (1)_{p_n}} \\ \cdot F_A^{(n)} (a+p_1+\dots+p_n, b_1+p_1, \dots, b_n+p_n; c_1+p_1, \dots, c_n+p_n; \\ x_1, \dots, x_n).$$

Multiplying both sides of (4.1) by $x_1^{\lambda_1-1} \dots x_n^{\lambda_n-1} y_1^{\mu_1-1} \dots y_n^{\mu_n-1}$ and applying the operators $D_{x_1}^{\delta_1} \dots D_{x_n}^{\delta_n} D_{y_1}^{\nu_1} \dots D_{y_n}^{\nu_n}$ on both sides, we get the result (3.1) after some simplification.

Similarly, results involving Lauricella functions $F_B^{(n)}$, $F_C^{(n)}$, $F_D^{(n)}$ may easily be extended.

Proof of (3.2) : The multiplication theorem for the Lauricella function $F_A^{(n)}$, given by H. Exton [1], is

$$(4.2) \quad F_A^{(n)} (a, b_1, \dots, b_n; c_1, \dots, c_n; x_1 y_1, \dots, x_n y_n) = \\ = \sum_{p_1, \dots, p_n=0}^{\infty} \frac{(a)_{p_1+\dots+p_n} (b_1)_{p_1} \dots (b_n)_{p_n} x_1^{p_1} (y_1-1)^{p_1} \dots x_n^{p_n} (y_n-1)^{p_n}}{(c_1)_{p_1} \dots (c_n)_{p_n} (1)_{p_1} \dots (1)_{p_n}} \\ \cdot F_A^{(n)} (a+p_1+\dots+p_n, b_1+p_1, \dots, b_n+p_n; c_1+p_1, \dots, c_n+p_n; \\ x_1, \dots, x_n)$$

Multiplying both sides of (4.2) by $x_1^{\lambda_1-1} y_1^{\mu_1-1} \dots x_n^{\lambda_n-1} y_n^{\mu_n-1}$ and applying the operators $D_{x_1}^{\delta_1} D_{y_1}^{\nu_1} \dots D_{x_n}^{\delta_n} D_{y_n}^{\nu_n}$ on both sides, we get, after some simplification, the result (3.2).

Similarly, the known results for Lauricella functions $F_B^{(n)}$, $F_C^{(n)}$ and $F_D^{(n)}$ may be extended.

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OBTAINING THE HORIZONTAL DEFLECTION OF A FALLING OBJECT WITHOUT THE KNOWLEDGE OF CORIOLIS FORCE

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ABSTRACT

The horizontal deflection relative to the rotating earth is determined of an object dropped from rest from the top of a tower. A salient feature of the present quest is that the relevant equations of motion are formed without bringing in the concept of coriolis force.

1. INTRODUCTION

The well-known problem of horizontal or eastward deflection of a body dropped from rest from a certain height has been worked out in many text-books of classical dynamics (see [2] and [4]) introducing the idea of coriolis force that arises due to the interaction between the rotation of the Earth and the velocity of the body as observed relative to the Earth. J. M. Potgieter [1] while solving the problem in an alternative method, first derived the equation of the path of the object with respect to an inertial frame (non-rotating) and then obtained the exact horizontal deflection by transforming the former equation to those with respect to a rotating frame relatively fixed to the earth. Stirling [3] derived the same result almost with the same technique as that of Potgieter [1] but only when the object is allowed to fall in the equatorial plane (zero latitude).

Potgieter [1] tackled the problem in unnecessary details incorporating many redundant equations. However, the present design brings forth altogether a different method of attacking the same problem.

2. SOLUTION TO THE PROBLEM

Case 1. Let the particle fall from the top of a tower of height h situated in the equatorial plane and (x, y) be the instantaneous position of the particle with reference to a non-rotating frame XOY , O being the centre of the earth. If g be the acceleration due to gravity, ω the angular velocity of the Earth about the polar axis and a its radius then the equations of motion of the particle are given by

$$\frac{d^2x}{dt^2} = -g \cos \omega t \quad (1)$$

$$\frac{d^2y}{dt^2} = -g \sin \omega t \quad (2)$$

along with the initial conditions that at $t = 0$, $x = a + h$, i. e. $y = 0$,

$$\frac{dx}{dt} = 0, \quad \frac{dy}{dt} = (a + h) \omega \text{ due to the earth's rotation} \quad \dots (3)$$

The solutions of equations (1) and (2) with the help of (3) can be obtained as

$$x = a + h - g (1 - \cos \omega t) / \omega^2 \quad (4)$$

$$y = (a + h) \omega t - g (-\sin \omega t) / \omega \quad (5)$$

which, on expanding $\cos(\omega t)$ and $\sin(\omega t)$ in series and neglecting the square and other higher terms of ω , reduce to the form

$$x = a + h - \frac{1}{2} g t^2 \quad \dots (6)$$

$$y = (a + h) \omega t - \frac{g \omega t^3}{6} \quad \dots \quad (7)$$

If T be the time taken by particle to reach the surface of the earth then at $t=T$, $x = a \cos(\omega T)$ and $y=d$ (say) so that by the use of eqns (6) and (7) along with the foregoing approximation we get :

$$h = \frac{1}{2} g T^2 \quad (8)$$

$$d = (a + h) \omega T - \frac{g \omega T^3}{6} \quad (9)$$

Eliminating T from (8), (9) and taking into consideration that the foot of the tower in the meantime has already moved eastward a distance equal to $a \sin(\omega T) \simeq a \omega T$, the eastward or horizontal deflection, from the foot of the tower, of the object is given by

$$D = d - a \omega T = -\frac{1}{3} g \omega T^3 = \frac{1}{3} 2h \omega \left(\frac{2h}{g} \right)^{\frac{1}{2}} \quad \dots \quad (10)$$

Case 2. If the particle be dropped from the top of a tower situated in the northern hemisphere at the latitude λ then because of the Earth's rotation about the polar axis, it initially acquires the velocity

$$\begin{aligned} \vec{V}_0 &= \vec{\omega} \times (a + h) \hat{r} \\ &= k \omega (a + h) (i \cos \lambda + k \sin \lambda) \\ &= j \omega (a + h) \cos \lambda \quad \dots \quad (11) \end{aligned}$$

where the right-handed inertial frame $OXYZ$ consists of the X -axis which is the line of intersection of the equatorial plane with the plane passing through the polar axis and the radius vector joining the centre

O of the Earth to the top of the tower, the Y -axis perpendicular to the former axis and the Z -axis coincident with the polar axis, (i, j, k) being the unit vectors associated with the system of axes. Equation (11) reveals that the particle initially gains a velocity equal to $\omega(a+h) \cos \lambda$ and that too in the eastward direction indicated by the Y -axis.

If (x, y, z) be the position of the particle at any instant of time t with respect to this non-rotating frame fixed in space, the equations of motion can be written as

$$\frac{d^2x}{dt^2} = -\frac{gx}{r}, \frac{d^2y}{dt^2} = -\frac{gy}{r}, \frac{d^2z}{dt^2} = -\frac{gz}{r} \quad \dots (12)$$

$$(r^2 = x^2 + y^2 + z^2)$$

where the initial conditions are

$$t=0, x=(a+h) \cos \lambda, y=0, z=(a+h) \sin \lambda,$$

$$\frac{dx}{dt} = \frac{dz}{dt} = 0 \text{ and } \frac{dy}{dt} = (a+h) \omega \cos \lambda \quad \dots (13)$$

It is a formidable task to obtain solutions to equations (12) subject to the initial conditions. However, approximate closed form solutions can be found out the equations, in the following way

If w is considered to be so small as to be totally negligible, obviously the path of the particle is vertical, i. e., one dimensional accounting for no deflection eastward or otherwise, which is tantamount to solving equations (12) wherein the parameters x, y, z as a way of the first approximation are replaced by their initial values from (13). Because of this first approximation, equations (12) reduce to the form

$$\frac{d^2x}{dt^2} = -g \cos \lambda, \frac{d^2y}{dt^2} = 0, \frac{d^2z}{dt^2} = -g \sin \lambda \quad \dots (14)$$

which, as a consequence of (13) and neglecting ω , yield

$$x = (a + h) \cos \lambda - \frac{1}{2} g \cos \lambda t^2$$

$$y = 0$$

$$z = (a + h) \sin \lambda - \frac{1}{2} g \sin \lambda t^2 \quad \dots \quad (15)$$

The second approximate solutions to equations (14), which can be carried out by not neglecting ω in (13), become

$$y = (a + h) (\omega \cos \lambda) t \quad (16)$$

where the expressions for x and z remain the same as in (16). In order to obtain the the third approximate solutions to the equations of set (12), we can approximate in its second equation y and r by $\omega (a + h) \cos \lambda t$ and $(a + h)$ respectively; in view of the second approximation (16), initial conditions (13), and ω^2 and other powers of ω being negligibly small, then, by virtue of (13), the integrations of (12) lead to

$$y = (a + h) \omega (\cos \lambda) t - (1/6) g \omega (\cos \lambda) t^3 \quad \dots \quad (17)$$

$$d = (a + h) \omega (\cos \lambda) T - (1/6) g \omega (\cos \lambda) T^3 \quad \dots \quad (18)$$

$$h = \frac{1}{2} g T^2 \quad \dots \quad (19)$$

where T is the time taken by the particle to fall on the surface of the earth. Combining equations (18) and (19) on the lines of (8) and (9), the eastward deflection of the particle relative to the foot of the tower can be obtained as

$$\begin{aligned} D &= d - a \omega (\cos \lambda) T \\ &= (1/3) g (\omega \cos \lambda) T^2 \\ &= (1/3) 2h (\cos \lambda) (2h/g)^{1/2} \quad \dots \quad (20) \end{aligned}$$

To continue this approximation process, we need to replace in the second equation of (12) y by its expression from (17) and r by its initial value as mentioned earlier, so as to obtain with the help of equations (13) and (19) the more accurate deflection.

$$D^1 = 1/3 \cdot 2h \omega \left[1 - \frac{h}{20(a+h)} \right] (\cos \lambda) \left(\frac{2h}{g} \right)^{1/2} \quad \dots \quad (21)$$

The comparison of (20) with (21) reveals that since $h \ll a$, the former gives the most ideal result. However, from the theoretical point of view, further ∞ continuation of this process entails greater accuracy in determining the deflection of the particle.

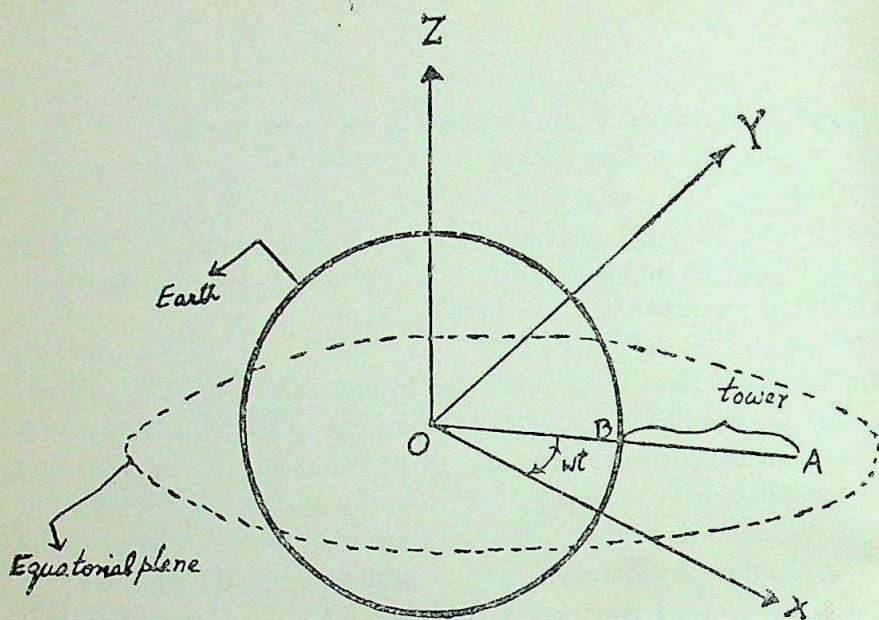


Fig 1. Horizontal deflection in case 1.

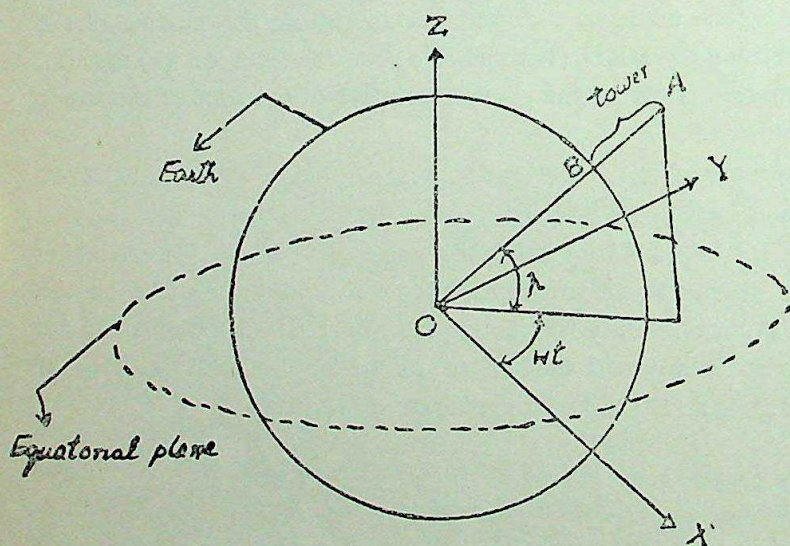


Fig 2. Horizontal deflection in case 2.

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ON EXPANSIONS OF q -SERIES OF THREE VARIABLES

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1. INTRODUCTION

Pathan [1] established certain expansions of hypergeometric functions of three variables. Recently, Singhal and his co-worker [2] pointed out that all the above expansions are immediate consequence of the simple binomial identity

$$(1.1) \quad \sum_{r=0}^n \binom{n}{r} (a)_{r+k} (c-a)_{n-r} = (a)_k (c+k)_n$$

and used it to establish expansions of certain hypergeometric functions of three variables due to Srivastava [4].

Here we make use of the following basic identity to show how it could be utilized to establish expansion of q -series of three variables, thus, extending the above known expansions to their basic analogues :

$$(1.2) \quad \sum_{r=0}^n \begin{bmatrix} n \\ r \end{bmatrix} [a]_{r+k} [c/a]_{n-r} a^{n-r} q^{\binom{n-r}{2}} = [a]_k [cq^k]_n,$$

where

$$\begin{bmatrix} n \\ r \end{bmatrix} = \frac{[q]_n}{[q]_r [q]_{n-r}}$$

and $[\alpha]_n = (1-\alpha)(1-\alpha q) \dots (1-\alpha q^{n-1})$, $[\alpha]_0 = 1$.

The proof of (1.2) is simple, and we omit the details.

Following Srivastava [5], we define a general q -hypergeometric series of three variables by

$$(1.3) \quad \phi^{(3)} \left[\begin{matrix} (a) : : (b); (b'); (b'') : (c); (c'); (c''); x, y, z \\ (e) : : (f); (f'); (f'') : (g); (g'); (g''); \lambda, \mu, \nu \end{matrix} \right]$$

$$= \sum_{m, n, p=0}^{\infty} \frac{[a]_{m+n+p} [b]_{m+n} [b']_{n+p} [b'']_{p-m} [c]_m [(c')]_n [(c'')]_{p-m}}{[e]_{m+n+p} [f]_{m+n} [f']_{n+p} [f'']_{p-m} [g]_m [(g')]_n [(g'')]_{p-m}}$$

$$\cdot \frac{[(c'')]_p x^m y^n z^p q^{\{\lambda m(m+1) + \mu n(n+1) + \nu p(p+1)\}/2}}{[(g'')]_p [q]_m [q]_n [q]_p}$$

where λ, μ and ν are positive to ensure rapid convergence of the series, In case $\lambda=0=\mu=\nu$, $|x| < 1, |y| < 1$ and $|z| < 1$.

Also, (a) shall stand for the sequence of A parameters $a_1, a_2, \dots, a_A$. In what follows, the other symbols will have their usual meaning.

2. EXPANSION FORMULAS

Here we show how (1.2) can be utilized to establish the q -series expansions. To show this, we establish the following expansion

$$(2.1) \quad \sum_{r=0}^m \sum_{s=0}^n \sum_{k=0}^p \begin{bmatrix} m \\ r \end{bmatrix} \begin{bmatrix} n \\ s \end{bmatrix} \begin{bmatrix} p \\ k \end{bmatrix} \frac{[u_r] [v/u]_{m-r} [u']_{n-s} [q/v']_s}{[v]_m [qu/v']_n [q/u'']_p}$$

$$\cdot [q/v'']_k [v''/u'']_{p-k} u^{m-r} (q/v')^{n-s} (q/v'')^{p-k}$$

$$\begin{aligned}
 & \cdot \phi^{(3)} \left[\begin{matrix} (a) : : (b); (b'); (b'') : uq^r, (c); u'q^{n-s}, (c'); u''q^{-n}, (c''); xq^{m-r}, \\ (e) : : (f); (f'); (f'') : vq^m, (g); v'q^{-s}, (g'); v''q^{-k}; \lambda, \mu, \nu \end{matrix} \right] \\
 & = \phi^{(3)} \left[\begin{matrix} (a) : : (b); (b'); (b'') : u, (c); u', (c'); n'', (c''); x, y, z \\ (e) : : (f); (f'); (f'') : v, (g'); v'', (g''); \lambda, \mu, \nu \end{matrix} \right].
 \end{aligned}$$

To prove (2.1), we replace the $\phi^{(3)}$ -series by its triple series given by (1.3), interchange the order of summations (valid under the given conditions), and evaluate the inner sums with the help of (1.2), and we get the right side of (2.1) after some simplification.

3. SPECIAL CASES

If we let $q \rightarrow 1$ in (2.1) after making some parametric changes, we get a known expansion due to Singhal and Soni [1, p. 38 (1.3)]. Thus, we can also establish the basic analogues of the other expansions due to Singhal and Soni [2]. These would, in turn, include the basic analogues of the expansions given earlier by Pathan [1].

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ON A CHAIN INVOLVING THE H -FUNCTION TRANSFORM

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ABSTACRT

In this paper we establish a new chain interconnecting a number of H -function transforms having different parameters. Full care has been taken to give an account of all the convergence and existance conditions for the validity of the chain. Since the H -function is very general in nature, the corresponding chains for a number of other integral transforms can easily be obtained as special cases of our chain. Similar chains for various simpler integral transforms have been obtained earlier by P. N. Rathie [7], U. C. Jain [6], S. P. Goyal [2], K. C. Gupta and P. K. Mittal [5],

1. INTRODUCTION

Throughout this work we use the notation

$$\phi(s) = H \left[f(t) : s : \begin{matrix} m, n & (a_i, \alpha_i)_{1,p} \\ p, q & (b_j, \beta_j)_{1,q} \end{matrix} \right]$$

$$= \int_0^\infty H_{p, q}^{m, n} \left[st \left| \begin{matrix} (a_j, \alpha_j)_{1, p} \\ (b_j, \beta_j)_{1, q} \end{matrix} \right. \right] f(t) dt \quad \dots (1.1)$$

for the H -function transform defined by Gupta and Mittal [4, p. 142]. The Kernel in (1.1) is the well-known H -function which is quite general in nature and includes most of the commonly used functions as its special cases. The definition of this H -function in terms of a Mellin Barnes contour integral, various conditions satisfied by its parameters for the convergence of the defining integral, some of its special cases, important properties, and asymptotic expansions, can be referred to in a standard work on the subject by Srivastava, Gupta, and Goyal [8, pp. 10-14]. It has been assumed that the various H -functions occurring in this paper satisfy the conditions corresponding appropriately to those given by (2.2.11) (on page 13 in the book referred to above. On account of the general nature of the kernel, this transform provides us with interesting unification and generalizations of a number of integral transforms (see [8, Chapter 4] for details).

2. MAIN THEOREM

THEOREM. If

$$\phi_1(s) = H \left[f(t) : s : \begin{matrix} m_1, n_1 \\ p_1, q_1 \end{matrix} : \begin{matrix} (a_j^{(1)}, \alpha_j^{(1)})_{1, p_1} \\ (b_j^{(1)}, \beta_j^{(1)})_{1, q_1} \end{matrix} \right], \quad \dots (2.1)$$

and

$$\phi_k(s) = H \left[t^{c_{k-1}} \phi_{k-1} \left(\frac{1}{t} \right) : s : \begin{matrix} m_k, n_k \\ p_k, q_k \end{matrix} : \begin{matrix} (a_j^{(k)}, \alpha_j^{(k)})_{1, p_k} \\ (b_j^{(k)}, \beta_j^{(k)})_{1, q_k} \end{matrix} \right] \quad \dots (2.2)$$

$$k = 2, 3, \dots, N),$$

then

$$\phi_N(s) = s^{-\sum_{i=1}^{N-1} c_i - (N-1)} \int_0^\infty H_{\sum_{i=1}^N m_i, \sum_{i=1}^N n_i} \left[st \begin{vmatrix} I_N \\ J_N \end{vmatrix} \right] f(t) dt \quad \dots (2.3)$$

where N is a positive integer greater than 1,

$$I_N = [(a_j^{(M)} + \{ \sum_{i=1}^{M-1} c_i + (M-1) \} \alpha_j^{(M)}, \alpha_j^{(M)})_{1, n_M}]_{M=1}^{N-1},$$

$$(a_j^{(N)} + \{ \sum_{i=1}^{N-1} c_i + (N-1) \} \alpha_j^{(N)}, \alpha_j^{(N)})_{1, p_N},$$

$$[(a_j^{(M)} + \{ \sum_{i=1}^{M-1} c_i + (M-1) \} \alpha_j^{(M)}, \alpha_j^{(M)})_{n_M+1, p_M}]_{M=1}^{N-1}$$

and

$$J_N = [(b_j^{(M)} + \{ \sum_{i=1}^{M-1} c_i + (M-1) \} \beta_j^{(M)}, \beta_j^{(M)})_{1, m_M}]_{M=1}^{N-1}.$$

$$(b_j^{(N)} + \{ \sum_{i=1}^{N-1} c_i + (N-1) \} \beta_j^{(N)}, \beta_j^{(N)})_{1, q_N},$$

$$[(b_j^{(M)} + \{ \sum_{i=1}^{M-1} c_i + (M-1) \} \beta_j^{(M)}, \beta_j^{(M)})_{m_M+1, q_M}]_{M=1}^{N-1},$$

it being assumed that the integrals involved in equations (2.1) and (2.2) are absolutely convergent and the following sets of conditions are satisfied:

$$A_N > 0, A'_N > 0, |\arg(s)| < \frac{1}{2} A'_N \pi,$$

$$\begin{aligned}
& \operatorname{Re} \left[\sum_{i=1}^{N-1} c_i + (N-1) \right] - \sum_{M=1}^{N-1} \max_{1 \leq j \leq n_M} \\
& \quad a_j^{(M)} + \left\{ \sum_{i=1}^{M-1} c_i + (M-1) \right\} \alpha_j^{(M)} - 1 \\
& \quad \left[\operatorname{Re} \frac{\quad}{a_j^{(M)}} \right] + \\
& + \min_{1 \leq j \leq n_N} \operatorname{Re} \frac{b_j^{(N)}}{\beta_j^{(N)}} > 0,
\end{aligned}$$

$$\begin{aligned}
& \operatorname{Re} \left[\sum_{i=1}^{N-1} c_i + (N-1) \right] - \sum_{M=1}^{N-1} \min_{1 \leq i \leq m_N} \\
& \quad b_j^{(M)} + \left\{ \sum_{i=1}^{M-1} c_i + (M-1) \right\} \beta_j^{(M)} \\
& \quad \left[\operatorname{Re} \frac{\quad}{\beta_j^{(M)}} \right] +
\end{aligned}$$

$$+ \max_{1 \leq j \leq n_N} \operatorname{Re} \frac{a_j^{(N)} - 1}{a_j^{(N)}} < 0,$$

where

$$A_N = \sum_{M=1}^{N-1} \left[\sum_{j=1}^{n_M} \alpha_j^{(M)} - \sum_{j=n_M+1}^{p_M} \alpha_j^{(M)} + \sum_{j=1}^{m_{(M)}} \beta_j^{(M)} - \sum_{j=m_M+1}^{q_M} \beta_j^{(M)} \right],$$

$$A'_N = \sum_{j=1}^{n_N} \alpha_j^{(N)} - \sum_{j=n_N+1}^{p_N} \alpha_j^{(N)} + \sum_{j=1}^{m_N} \beta_j^{(N)} - \sum_{j=m_N+1}^{q_N} \beta_j^{(N)}$$

and the notation $[(a_j^{(M)}, \alpha_j^{(M)})_{1, u_M}]_{M=1}^N$ stands for a sequence of

$$\sum_{i=1}^N u_i \text{ pairs } (a_j^{(1)}, \alpha_j^{(1)})_{1, u_1}, (a_j^{(2)}, \alpha_j^{(2)})_{1, u_2}, \dots, (a_j^{(N)}, \alpha_j^{(N)})_{1, u_n}.$$

Proof. Interpreting (2.2) for $k=2$ with the help of (1.1), we get

$$\phi_2(s) = \int_0^\infty x^{c_1} H_{p_2, q_2}^{m_2, n_2} \left[s x \left| \begin{matrix} (a_j^{(2)}, \alpha_j^{(2)})_{1, p_2} \\ (b_j^{(2)}, \beta_j^{(2)})_{1, q_2} \end{matrix} \right. \right] \phi_1\left(\frac{1}{x}\right) dx \quad \dots (2.4)$$

Substituting the value of $\phi_1(1/x)$ from (2.1) in (2.4), we get

$$\phi_2(s) = \int_0^\infty x^{c_1} H_{p_2, q_2}^{m_2, n_2} \left[s x \left| \begin{matrix} (a_j^{(2)}, \alpha_j^{(2)})_{1, p_2} \\ (b_j^{(2)}, \beta_j^{(2)})_{1, q_2} \end{matrix} \right. \right] \cdot \left\{ \int_0^\infty H_{p_1, q_1}^{m_1, n_1} \left[t x^{-1} \left| \begin{matrix} (a_j^{(1)}, \alpha_j^{(1)})_{1, p_1} \\ (b_j^{(1)}, \beta_j^{(1)})_{1, q_1} \end{matrix} \right. \right] f(t) dt \right\} dx \quad \dots (2.5)$$

On inverting the order of integrations in (2.5) which is easily seen to be permissible under the conditions stated with the theorem we get

$$\phi_2(s) = \int_0^\infty \left\{ \int_0^\infty x^{c_1} H_{p_2, q_2}^{m_2, n_2} \left[t x^{-1} \left| \begin{matrix} (a_j^{(2)}, \alpha_j^{(2)})_{1, p_2} \\ (b_j^{(2)}, \beta_j^{(2)})_{1, q_2} \end{matrix} \right. \right] dx \right\} f(t) dt \quad \dots (2.6)$$

Evaluating the x -integral involved in (2.6) with the help of modified form of a result [3, p. 601], we get

$$\phi_2(s) = \int_0^\infty s^{-c_1-1} H_{\substack{m_1+m_2, n_1+n_2 \\ p_1+p_2, q_1+q_2}} \left[ts \begin{vmatrix} I_2 \\ J_2 \end{vmatrix} \right] f(t) dt \quad \dots (2.7)$$

where

$$I_2 = [(a_j^{(1)}, \alpha_j^{(1)})_{1, n_1}, (a_j^{(2)} + (c_1+1) \alpha_j^{(2)}, \alpha_j^{(2)})_{1, p_2}, \\ (a_j^{(1)}, \alpha_j^{(1)})_{n_1+1, p_1}]$$

and

$$J_2 = [(b_j^{(1)}, \beta_j^{(1)})_{1, m_1}, (b_j^{(2)} + (c_1+1) \beta_j^{(2)}, \beta_j^{(2)})_{1, q_2}, \\ (b_j^{(1)}, \beta_j^{(1)})_{m_1+1, q_1}]$$

and the following set of conditions are assumed to be satisfied :

$$A_2 > 0, A'_2 > 0, | \arg(s) | < \frac{1}{2} A'_2 \pi$$

$$\operatorname{Re}(c_1+1) - \max_{1 \leq j \leq m_1} \left[\operatorname{Re} \frac{a_j^{(1)}-1}{a_j^{(1)}} \right] + \min_{1 \leq j \leq m_2} \left[\operatorname{Re} \frac{b_j^{(2)}}{\beta_j^{(2)}} \right] > 0,$$

$$\operatorname{Re}(c_1+1) - \min_{1 \leq j \leq m_1} \left[\operatorname{Re} \frac{b_j^{(1)}}{\beta_j^{(1)}} \right] + \max_{1 \leq j \leq n_2} \left[\operatorname{Re} \frac{a_j^{(2)}-1}{\alpha_j^{(2)}} \right] < 0,$$

where

$$A_2 = \sum_{j=1}^{n_1} \alpha_j^{(1)} - \sum_{j=n_1+1}^{p_1} \alpha_j^{(1)} + \sum_{j=1}^{m_1} \beta_j^{(1)} - \sum_{j=m_1+1}^{q_1} \beta_j^{(1)}$$

and

$$A'_2 = \sum_{j=1}^{n_2} \alpha_j^{(2)} - \sum_{j=n_2+1}^{p_2} \alpha_j^{(2)} + \sum_{j=1}^{m_2} \beta_j^{(2)} - \sum_{j=m_2+1}^{q_2} \beta_j^{(2)}.$$

Now, by assuming the result (2.3) to be true for $N=K$ and proceeding in a manner similar to the above, we can easily arrive at the corresponding result for $N=K+1$. Much of the computational work involved in proving this is straightforward, so we omit the details. Also, from (2.1) and (2.7), we find that the result (2.3) is true for $N=2$. Thus by mathematical induction, we can say that the result (2.3) is true for every positive integer N greater than 1.

3. SPECIAL CASES

1. If in the main theorem, we replace m_k by m_k+1 , p_k by m_k, q_k by m_k+1 , n_k by 0, $a_j^{(k)}$ by $a_j^{(k)}+b_j^{(k)}$ ($j=1,2,\dots,m_k, k=1,2,\dots,N$) and take $\alpha_j^{(k)}=\beta_j^{(k)}=1$ ($j=1,2,\dots,m_k+1; k=1,2,\dots,N$), $b^{(k)}_{m_k+1}=\rho_k$ therein, it reduces to the following result involving the G -function transforms (see [1] and [8, chapter 4]) :

COROLLARY 1. *If*

$$\phi_1(s) = G \left[f(t) : s : \begin{matrix} m_1+1, 0 \\ m_1, m_1+1 \end{matrix} : \begin{matrix} (a_j^{(1)}+b_j^{(1)})_{1, m_1} \\ (b_j^{(1)})_{1, m_1}, \rho_1 \end{matrix} \right] \dots (3.1)$$

and

$$\phi_k(s) = G \left[t^{c_{k-1}} \phi_{k-1} \left(\frac{1}{t} \right) : s : \begin{matrix} m_k+1, 0 \\ m_k, m_k+1 \end{matrix} : \begin{matrix} (a_j^{(k)}+b_j^{(k)})_{1, m_k} \\ (b_j^{(k)})_{1, m_k}, \rho_k \end{matrix} \right] \dots (3.2)$$

$$(k=2, 3, \dots, N),$$

then

$$g_N(s) = s^{-\sum_{i=1}^{N-1} c_i - (N-1)} \int_0^\infty G \begin{matrix} \sum_{i=1}^N m_i + N, 0 \\ \sum_{i=1}^N m_i, \sum_{i=1}^N m_i + N \end{matrix} \begin{bmatrix} st & P_N \\ & Q_N \end{bmatrix} f(t) dt \quad \dots (3.3)$$

where

$$P_N = \left[\left(a_j^{(M)} + b_j^{(M)} + \sum_{i=1}^{M-1} c_i + M - 1 \right)_{1, m_M} \right]_{M=1}^N,$$

$$Q_N = \left[\left(b_j^{(M)} + \sum_{i=1}^{M-1} c_i + M - 1 \right)_{1, m_M} \right]_{M=1}^N,$$

$$\left[\left(\rho_M + \sum_{i=1}^{M-1} c_i + M - 1 \right) \right]_{M=1}^N$$

and the conditions of validity directly obtainable from the main theorem are assumed to be satisfied.

II. If, in Corollary 1, we take $m_k = \rho_k = 0$ ($k = 1, 2, \dots, N$), it reduces to the following result involving the well-known Laplace transforms :

COROLLARY 2. *If*

$$\phi_1(s) = L[f(t); s]$$

and

$$\phi_k(s) = L \left[t^{C_{k-1}} \phi_{k-1} \left(\frac{1}{t} \right); s \right], \quad (k = 2, 3, \dots, N),$$

... (3.5)

then

$$\phi_N(s) = s^{-\sum_{i=1}^{N-1} c_i - (N-1)} \int_0^\infty H_{0, N}^{N, 0} \left[s t \middle| \begin{matrix} - \\ R_N \end{matrix} \right] f(t) dt \quad \dots (3.6)$$

where $R_N = \left(\sum_{i=1}^{M-1} c_i + M-1 \right) \frac{N}{M-1}$ and the conditions of validity

directly obtainable from the main theorem are assumed to be satisfied,

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APPLICATION OF GENERALISED BESSEL'S PARTIAL DIFFERENTIAL OPERATORS TO PARTIAL DIFFERENTIAL EQUATIONS

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ABSTRACT

In Section I we have derived some operational formulas with the help of the partial differential operators of generalised Bessel polynomials which may be regarded as the generators of Lie algebra. In section 3 some applications of operational formulas are also pointed out.

1. INTRODUCTION

Isaac I. H. Chen and T. W. Barrett [1] have used Bessel's ordinary differential operators (which raise and lower the index of Bessel's function of the first kind) to solve some second-order linear ordinary differential equations. It may be pointed out that the operators used by Chen and Barrett are not proper Lie elements in order to generate a Lie algebra. The object of the present paper is to use the partial differential operators of M. - P. Chen and C. - C. Feng [2] in connection with the generalised Bessel polynomials, which are regarded as generators Lie algebra, in the derivation of some operational results and finally in the solution of those partial differential equations which can be factorized by means of the generators of the Lie algebra for the generalised Bessel polynomials.

In a recent paper [3] we have explained the reason why it is impossible to obtain generators of Lie algebra for generalised Bessel polynomials with two operators which raise and lower the index of generalised Bessel polynomials by unity.

In fact, from [2] we have

$$(1.1) \quad \begin{cases} [xy \frac{\partial}{\partial x} + y^2 \frac{\partial}{\partial y} + (n-1)y] x_n^\alpha = (n + \alpha - 1) x_n^{\alpha+1} \\ [x^2 y^{-1} \frac{\partial}{\partial x} - (nx - \beta) y^{-1}] x_n^\alpha = \beta x_n^{\alpha-1}, \end{cases}$$

where $x_n^\alpha = y_n^\alpha(x) y^\alpha$, $y_n^\alpha(x)$ is the generalised Bessel polynomial

(cf., e.g., [4, p. 75]).

If we put

$$(1.2) \quad \begin{cases} R = xy \frac{\partial}{\partial x} + y^2 \frac{\partial}{\partial y} + (n-1)y \\ L = x^2 y^{-1} \frac{\partial}{\partial x} - (nx - \beta) y^{-1} \end{cases}$$

then

$$(1.3) \quad \begin{pmatrix} 0 & R \\ L & 0 \end{pmatrix} \begin{pmatrix} x_n^{\alpha+1} \\ x_n^\alpha \end{pmatrix} = \begin{pmatrix} (n + \alpha - 1) x_n^{\alpha+1} \\ \beta x_n^\alpha \end{pmatrix}$$

where

$$x_n^\alpha = y_n^\alpha(x) y^\alpha.$$

Also we notice that

$$(1.4) \quad LR [y_n^\alpha (x) y^\alpha] = \beta (n + \alpha - 1) y_n^\alpha (x) y^\alpha,$$

which yields the well-known relation

$$(1.5) \quad [x^2 \frac{\partial^2}{\partial x^2} + xy \frac{\partial^2}{\partial x \partial y} + \beta \frac{\partial}{\partial x} - ny \frac{\partial}{\partial y} - n(n-1)] y_n^\alpha (x) y^\alpha = 0$$

Again

$$[R, L] = -\beta$$

where

$$[R, L] = RL - LR$$

Also we have

$$(1.6) \quad yL = x^2 \frac{\partial}{\partial x} - (nx - \beta)$$

$$xy^{-1} R - yL = xy \frac{\partial}{\partial y} - (\beta - x)$$

2. Derivation of Operational Formulas

Consider the partial differential equation

$$(2.1) \quad xy \frac{\partial u}{\partial x} + y^2 \frac{\partial u}{\partial y} + (n-1)yu = f(x, y)$$

which is equivalent to

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + (n-1)u = y^{-1} f(x, y).$$

Hence the corresponding system of ordinary differential equation is

$$(2.2) \quad \frac{dx}{x} = \frac{dy}{y} = \frac{du}{y^{-1} f(x, y) - (n-1)u}$$

Solving (2.2) we get $xy^{-1} = c_1$ and

$$(2.3) \quad ux^{n-1} = c_1 \int x^{n-3} f(x, c_1^{-1}x) dx + c_2$$

$$= \Psi_1(x, c_1) + c_2, \quad (\text{say}).$$

Hence

$$(2.4) \quad \frac{1}{R} \left[f(x, y) \right] = x^{1-n} \Psi_1(x, c_1) \Big|_{c_1 = xy^{-1}} + x^{1-n} \phi_1(xy^{-1}).$$

where Ψ_1 is given in (2.3) and ϕ_1 is arbitrary.

Corollary 1. If $y^{-1}f(x, y)$ = a function of x only = $P(x)$, (say), then

$$(2.5) \quad \frac{1}{R} \left[f(x, y) \right] = x^{1-n} \int x^{n-2} P(x) dx + x^{1-n} \phi_1(xy^{-1}),$$

Where ϕ_1 is arbitrary.

Corollary 2. If $y^{-1}f(x, y)$ = a function of y only = $Q(y)$, (say), then

$$(2.6) \quad \frac{1}{R} \left[f(x, y) \right] = y^{1-n} \int y^{n-2} Q(y) dy + y^{1-n} \phi_1(xy^{-1}),$$

where ϕ_1 is arbitrary.

Next, consider the partial differential equation

$$(2.7) \quad x^2 y^{-1} \frac{\partial u}{\partial x} - (nx - \beta) y^{-1} u = F(x, y),$$

which is equivalent to

$$x^2 \frac{\partial u}{\partial x} - (nx - \beta) u = y F(x, y).$$

Hence the corresponding system of ordinary differential equation is

$$(2.8) \quad \frac{dx}{x^2} = \frac{dy}{0} = \frac{du}{yF(x, y) + (nx - \beta)u}$$

Solving (2.8), we get $y = c$, and

$$(2.9) \quad u(e^{-\beta/x} x^{-n}) = c_1 \int \frac{F(x, c_1) dx}{x^{n+2} e^{\beta/x}} + c_2$$

$$= \Psi_2(x, c_1) + c_2 \quad (\text{say})$$

Hence

$$(2.10) \quad \frac{1}{L} F(x, y) = e^{\beta/x} x^n \Psi_2(x, c_1) \Big|_{c_1=y} + e^{\beta/x} x^n \phi_2(y)$$

where Ψ_2 is given in (2.9) and ϕ_2 is arbitrary.

COROLLARY 8. If $yF(x, y)$ = a function of x only = $P(x)$, (say), then

$$(2.11) \quad \frac{1}{L} F(x, y) = e^{\beta/x} x^n \int \frac{P(x)}{x^{n+2} e^{\beta/x}} dx + e^{\beta/x} x^n \phi_2(y)$$

where ϕ_1 is arbitrary.

COROLLARY 4. If $yF(x, y)$ = a function of y only = $Q(y)$, (say), then there exists no solution of u .

3. Application of the Operational Formulas

Consider the partial differential equation

$$(3.1) \quad x^3 \frac{\partial^2 u}{\partial x^2} + x^2 y \frac{\partial^2 u}{\partial x \partial y} + \beta x \frac{\partial u}{\partial x} - (nx - \beta) y \frac{\partial u}{\partial y} - \{n(nx - \beta) - (nx - 2\beta)\} u = f(x, y).$$

Since

$$\begin{aligned} & [xy \frac{\partial}{\partial x} + y^2 \frac{\partial}{\partial y} + (n-1)y] [x^2 y^{-1} \frac{\partial}{\partial x} - (nx-\beta) y^{-1}] \\ &= x^3 \frac{\partial^2}{\partial x^2} + x^2 y \frac{\partial^2}{\partial x \partial y} + \beta x \frac{\partial}{\partial x} - (nx-\beta) y \frac{\partial}{\partial y} - \{ n\{nx-\beta\} - (nx-2\beta) \}. \end{aligned}$$

The above equation (3.1) becomes

$$RLu = f(x, y).$$

It follows from (2.4) that

$$(3.2) \quad Lu = x^{1-n} \Psi_1(x, c_1) \Big|_{c_1 = xy^{-1}} + x^{1-n} \phi_1(xy^{-1})$$

$$(3.3) \quad = F(x, y), \text{ (say),}$$

where Ψ_1 is given in (2.3) and ϕ_1 is arbitrary. Again it follows from (2.10) that

$$(3.4) \quad u = e^{\beta/x} x^n \Psi_2(x, c_1) \Big|_{c_1=y} + e^{\beta/x} x^n \phi_2(y)$$

where Ψ_2 is given in (2.9) and ϕ_2 is arbitrary.

On the other hand, if we consider the equation

$$LRu = F(x, y),$$

which is equivalent to

$$\begin{aligned} (3.5) \quad x^3 \frac{\partial^2 u}{\partial x^2} + x^2 y \frac{\partial^2 u}{\partial x \partial y} + \beta x \frac{\partial u}{\partial x} - (nx-\beta) y \frac{\partial u}{\partial y} \\ - (nx-\beta)(n-1)u = F(x, y), \end{aligned}$$

then it follows from (2.10) that

$$(3.6) \quad Ru = e^{\beta/x} x^n \Psi_2(x, c_1) \Big|_{c_1=y} + e^{\beta/x} x^n \phi_2(y)$$

$$(3.7) \quad = f(x, y), \text{ say,}$$

where Ψ_2 is given in (2.9) and ϕ_2 is arbitrary. Hence, from (2.4) it follows that

$$(3.8) \quad u = x^{1-n} \Psi_1(x, c_1) \Big|_{c_1=xy^{-1}} + x^{1-n} \phi_1(xy^{-1})$$

where Ψ_1 is given in (2.3) and ϕ_1 is arbitrary.

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FIXED POINT THEOREM FOR MULTIVALUED MAPPINGS UNDER THE CARISTI - KIRK TYPE CONDITION

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ABSTRACT

In this paper we prove a fixed point theorem for a pair of multivalued mappings under the Caristi-Kirk type condition. We also give the condition for the existence of such a fixed point.

1. INTRODUCTION

S. B. Nadler [2] proved a fixed point theorem for multivalued contraction mappings. We use the following notations and definitions as given by him in [2] for our work.

Definitions 1. Let (X, d) be a metric space, then

(i) $CB(X) = \{C : C \text{ is a nonempty closed and bounded subset of } X\}$

(ii) $N(\epsilon, A) = \{x \in X \mid d(x, y) < \epsilon \text{ for some } y \in A\}.$

$\epsilon > 0$ and $A \in CB(X).$

(iii) $H(A, B) = \inf \{A \subset N(\epsilon, B) \text{ and } B \subset N(\epsilon, A)\},$

$\epsilon > 0$ and $A, B \in CB(X).$

The function H is a metric for $CB(X)$ called the Hausdorff metric for $CB(X)$.

Let (X, d_1) and (Y, d_2) be two metric spaces. A function $F: X \rightarrow CB(Y)$ is said to be a multivalued contraction mapping of X into Y if

$$H(F(x), F(y)) \leq \alpha d_1(x, y) \quad \forall x, y \in X,$$

$$0 \leq \alpha < 1$$

A point x is said to be a fixed point of a multivalued mapping F if $x \in F(x)$.

We shall also make use of the following lemmas, which are noted implicitly in Nadler [2]

Lemma I. If $A, B \in CB(X)$ and $\bar{x} \in A$, then, for each positive number η , there exists a $y \in B$ such that

$$d(\bar{x}, y) \leq H(A, B) + \eta.$$

Lemma 2. Let $\{A_n\}$ be a sequence of sets in $CB(X)$, and suppose $\lim H(A_n, A_0) = 0$, where $A_0 \in CB(X)$. Then, if $x_n \in A_n$, $n = 1, 2, 3, \dots$, and $\lim x_n = x_0$, it follows that $x_0 \in A_0$.

Recently, Caristi ([1], Theorem 2.1) presented a generalization of the contraction mapping principle which is now known as Caristi-Kirk fixed point theorem. The following proposition is observed by Felix Browder (See Caristi and Kirk [2]) as the modified version of the famous Caristi-Kirk theorem:

Proposition. Let (X, d) be a complete metric space and $g: X \rightarrow X$ continuous. Suppose that there exists a lower semicontinuous function $\phi: X \rightarrow \mathbb{R}^+$ such that

$$d(x, gx) \leq \phi(x) - \phi(g(x)), x \in X.$$

Then, for any $x \in X$, the sequence $\{g^n(x)\}$ converges to a fixed point at g .

We prove

Theorem 1. Let (X, d) be a complete metric space and F_1, F_2 be two continuous multivalued mappings on X . Let there be two mappings ϕ and Ψ on $CB(X) \rightarrow [0, \infty)$ such that

$$(1.1) \quad H(F_1(x), F_2(y)) \leq \phi(x) - \phi(F_1(x)) + \Psi(y) - \Psi(F_2(y))$$

for all $x, y \in X$.

$$(1.2) \quad \phi(A) \leq \phi(B) \text{ and } \Psi(A) \leq \Psi(B), A, B \text{ in } CB(X), \text{ and } A \subset B.$$

Then F_1 and F_2 have a common fixed point. Further, if any of the multivalued mappings is one-one, then this common fixed point is a unique common fixed point of the one-one mapping :

Remark 1. We will use $\phi(x)$ instead of $\phi(\{x\})$ for the singleton set $\{x\}$, $x \in X$.

Proof. Let $x_0, y_0 \in X$ and $x_1 \in F_1(x_0)$. By Lemma 1, $\exists$

a $y_1 \in F_2(y_0)$ such that

$$d(x_1, y_1) < H(F_1(x_0), F_2(y_0)) + \frac{1}{2}$$

$$\leq \phi(x_0) - \phi(F_1(x_0)) + \Psi(y_0) - \Psi(F_2(y_0)) + \frac{1}{2} \text{ by (1.1)}$$

Since $x_1 \in F_1(x_0)$, i. e., $\{x_1\} \subset F_1(x_0)$ by (1.2), we have

$$\phi(x_1) \leq \phi(F_1(x_0)). \text{ Similarly, } \Psi(y_1) \leq \Psi(F_2(y_0))$$

Hence

$$d(x_1, y_1) < \phi(x_0) - \phi(x_1) + \Psi(y_0) - \Psi(y_1) + \frac{1}{2}$$

Since $y_1 \in F_2(y_0)$, $\exists$ an $x_2 \in F_1(x_1)$ such that

$$\begin{aligned} d(x_2, y_1) &< H(F_1(x_1), F_2(y_0)) + \frac{1}{2} \\ &\leq \phi(x_1) - \phi(F_1(x_1)) + \Psi(y_0) - \Psi(F_2(y_0)) + \frac{1}{2} \\ &\leq \phi(x_1) - \phi(x_2) + \Psi(y_0) - \Psi(y_1) + \frac{1}{2} \end{aligned}$$

Since $x_2 \in F_1(x_1)$, $\exists$ a $y_2 \in F_2(y_1)$ such that

$$d(x_2, y_2) < \phi(x_1) - \phi(x_2) + \Psi(y_1) - \Psi(y_2) + \frac{1}{2^2}$$

and, since $y_2 \in F_2(y_1)$, $\exists$ an $x_3 \in F_1(x_2)$ such that

$$d(x_3, y_3) < \phi(x_2) - \phi(x_3) + \Psi(y_1) - \Psi(y_2) + \frac{1}{2^2}$$

proceeding in this manner, we obtain $\{x_n\}$ and $\{y_n\}$ such that

$$x_n \in F_1(x_{n-1}), y_n \in F_2(y_{n-1}) \text{ and}$$

$$d(x_{n+1}, y_n) < \phi(x_n) - \phi(x_{n+1}) + \Psi(y_{n-1}) - \Psi(y_n) + \frac{1}{2^n}$$

So, for any positive integer n , we have

$$\begin{aligned} (1.3) \quad \sum_{r=1}^n d(x_r, y_r) &\leq \sum_{r=1}^n \{ \phi(x_{r-1}) - \phi(x_r) + \Psi(y_{r-1}) - \Psi(y_r) \} \\ &\quad + \sum_{r=1}^n \frac{1}{2^r} \end{aligned}$$

$$< \phi(x_0) - \phi(x_n) + \Psi(y_0) - \Psi(y_n) + 1$$

$$\leq \phi(x_0) + \Psi(y_0) + 1, \text{ since } \phi(x_n) \geq 0,$$

$$\Psi(y_n) \geq 0.$$

Similarly,

$$(1.4) \quad \sum_{r=1}^n d(x_r, x_{r+1}) \leq \sum_{r=1}^n [d(x_r, y_r) + d(x_{r+1}, y_r)]$$

$$< \phi(x_0) + \phi(x_1) + 2\psi(y_0) + 2, \text{ and}$$

$\sum_{n=1}^{\infty} d(x_n, x_{n+1})$ is convergent. So $d(x_n, x_{n+1}) \rightarrow 0$

as $n \rightarrow \infty$. Thus $\{x_n\}$ is a Cauchy sequence and converges to $z_1 \in X$. Since F_1 is continuous and $x_n \in F_1(x_{n-1})$ for $n=1, 2, 3, \dots$, by Lemma 2, z_1 is a fixed point of F_1 .

Now

$$\sum_{r=1}^n d(y_r, y_{r+1}) \leq \sum_{r=1}^n [d(x_{r+1}, y_r) + d(x_{r+1}, y_{r+1})]$$

$$< 2\phi(x_1) + \psi(y_0) + \psi(y_1) + 2.$$

Therefore, $\{y_n\}$ is a Cauchy sequence and converges to $z_2 \in X$, so that, by Lemma 2, z_2 is a fixed point of F_2 . By (1.3),

$\sum_{n=1}^{\infty} d(x_n, y_n)$ is convergent and $\rightarrow 0$ as $n \rightarrow \infty$

$$\text{Now } d(z_1, z_2) \leq d(z_1, x_n) + d(x_n, y_n) + d(y_n, z_2).$$

Taking limit of both sides as $n \rightarrow \infty$, we have

$$d(z_1, z_2) = 0 \text{ and } z_1 = z_2.$$

Let F_1 be one-one and w be any fixed point of F_1 . Suppose $z_1 = z_2 = z$. Then

$$H(F_1(w), F_2(z)) \leq \phi(w) - \phi(F_1(w)) + \psi(z) - \psi(F_2(z)).$$

Since $w \in F_1(w)$ and $z \in F_2(z)$, by (1.2)

$$\phi(w) \leq \phi(F_1(w)) \text{ and } \psi(z) \leq \psi(F_2(z)). \quad \text{Hence}$$

$H(F_1(w), F_2(z)) = 0$, so that $F_1(w) = F_2(z)$. Since this is true for any fixed point w of F_1 , and z is a fixed point of F_1 , we have

$$F_1(z) = F_2(z).$$

Thus $F_1(w) = F_1(z)$ and since F_1 is one-one, $w = z$. This completes the proof.

If both F_1 and F_2 are one-one, this common fixed point would be a unique common fixed point of both F_1 and F_2 .

Corollary. Let (X, d) be a complete metric space and $T = \{F_\lambda : \lambda \in \Lambda\}$ be a family of continuous multivalued mappings on X . Suppose that for each $\lambda \in \Lambda$ there exists ϕ_λ on $CB(X) \rightarrow [0, \infty)$ satisfying

$$(1.5) \quad \phi_\lambda(A) \leq \phi_\lambda(B) \quad \forall A, B \text{ in } CB(X) \text{ with } A \subset B.$$

(1.6) Let $\exists$ a mapping $F_\alpha \in T$ which is one-one and for any other $F_\beta \in T$ ($\beta \neq \alpha$) we have

$$H(F_\alpha(x), F_\beta(y)) \leq \phi_\alpha(x) - \phi_\alpha(F_\alpha(x)) + \phi_\beta(y) - \phi_\beta(F_\beta(y))$$

for all x, y in X .

Then the mapping of T has a common fixed point which is the unique fixed point of F ,

Remark 2. Theorem 1 is a generalization of the above proposition due to Felix Browder. In fact, by putting $F_1 = I_x$, $F_2 = g$, $x=y$ in Theorem 1, we obtain the above proposition.

2. A FURTHER RESULT

Now we present a result giving the condition for the existence of such a fixed point, considering a multivalued mapping.

Theorem 2. Let (X, d) be a metric space, F be a multivalued mapping on X and ϕ be a mapping on $CB(X) \rightarrow [0, \infty)$ such that

(2.1) $\phi(A) \leq \phi(B) \forall A, B$ in $CB(X)$ with $A \subset B$.

Suppose $z \in X$ be such that

$$(2.2) \quad H(F(x), z) \leq \phi(x) - \phi(F(x)) \quad \forall x \text{ in } X$$

then z is a fixed point of F . If F is one-one, then z is the unique fixed point of F .

Proof. Let $x_0 \in X$, then by Lemma 2, there exists an $x_1 \in F(x_0)$ such that

$$\begin{aligned} d(x_1, z) &< H(F(x_0), z) + \frac{1}{2} \\ &\leq \phi(x_0) - \phi(F(x_0)) + \frac{1}{2}, \text{ by (2.2) } . \\ &\leq \phi(x_0) - \phi(x_1) + \frac{1}{2}, \text{ by (2.1) } \end{aligned}$$

Again, there exists an $x_2 \in F(x_1)$ such that

$$\begin{aligned} d(x_2, z) &\leq H(F(x_1), z) + \frac{1}{2^2} \\ &\leq \phi(x_1) - \phi(x_2) + \frac{1}{2} \end{aligned}$$

proceeding in this manner, we construct a sequence $\{x_n\}$ such that $x_n \in F(x_{n-1})$, and

$$d(x_n, z) \leq \phi(x_{n-1}) - \phi(x_n) + \frac{1}{2^n}, \quad n = 1, 2, 3, \dots$$

Then

$$\sum_{r=1}^n d(x_r, z) \leq \sum_{r=1}^n \left\{ \phi(x_{r-1}) - \phi(x_r) + \frac{1}{2^r} \right\} \leq \phi(x_0) + 1$$

Hence the series $\sum_{n=1}^{\infty} d(x_n, z)$ is convergent so that

$d(x_n, z) \rightarrow 0$ as $n \rightarrow \infty$. Hence $x_n \rightarrow z$ as $n \rightarrow \infty$. Since

F is continuous and $x_n \in F(x_{n-1})$, $n=1, 2, 3, \dots$, by Lemma 2, z is a fixed point of F .

Let F be one-one and z be any fixed point of F . Then

$$H(F(z_1), z) \leq \phi(z_1) - \phi(F(z)) \leq 0, \text{ so that}$$

$F(z_1) = \{z\}$ This being these of any fixed point of F , we have

$F(z) = \{z\} = F(z_1)$, F being one-one, $z = z_1$. This completes the proof of Theorem 2.

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A STABILITY CRITERION FOR A MORE DENSE STREAMING FLUID OVER A LESS DENSE FLUID IN POROUS MEDIUM

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ABSTRACT

A criterion for the stability of a more dense fluid layer of finite thickness flowing over a less dense fluid of infinite extent with a relative horizontal velocity in a porous medium has been obtained ,

1. INTRODUCTION

The instability of two uniform fluids in relative horizontal motion separated by a horizontal boundary, when the less dense fluid overlies the more dense one, has been treated by Chandrasekhar [1]. The instability of streaming fluids in a porous medium has been considered by Sharma and Spanos [2]. In these studies, both the fluids are assumed to be of infinite extent and that the density of the upper fluid is assumed to be less than the density of the lower fluid .

When a more dense fluid overlies a less dense one in the absence of streaming, there will always be instability [1]. But when a more

dense fluid layer of finite thickness (small) is streaming over a less dense fluid of infinite extent in porous medium one may expect stability also. Situations of more dense fluids streaming over the less dense ones in a porous medium are of frequent occurrence (e. g. water over oil) below the surface of the Earth. The stability of such configurations is of much importance to fluid dynamicists and geophysicists.

The present note is an attempt in this direction wherein we discuss the stability of a more dense fluid layer of finite thickness flowing over a less dense fluid of infinite extent with a relative horizontal velocity in porous medium.

2. THE STABILITY CRITERION AND DISCUSSION

The initial stationary state whose stability we wish to examine is that of an incompressible fluid in which there is horizontal streaming in the x -direction with a velocity $U(z)$ through a homogeneous porous medium. Let $\delta\rho$, δp , u_i (u, v, w) and δz_s denote the perturbations in density ρ , pressure p , velocity U_i ($U(z), 0, 0$) and surfaces of separation z_s respectively; g, μ, k_1, ε and T_s stand respectively for acceleration due to gravity, fluid viscosity, medium permeability, medium porosity and pseudo (effective) interfacial tensions T_s at some prescribed levels z_s . Then the linearized perturbation equations are :

$$(1) \quad \frac{\rho}{\varepsilon} \left(\frac{\partial u}{\partial t} + \frac{U}{\varepsilon} \frac{\partial u}{\partial x} + \frac{\omega}{\varepsilon} \frac{dU}{dz} \right) = - \frac{\partial}{\partial x} \delta p - \frac{\mu}{k_1} u,$$

$$(2) \quad \frac{\rho}{\varepsilon} \left(\frac{\partial v}{\partial t} + \frac{U}{\varepsilon} \frac{\partial v}{\partial x} \right) = - \frac{\partial}{\partial y} \delta p - \frac{\mu}{k_1} v,$$

$$(3) \quad \frac{\rho}{\varepsilon} \left(\frac{\partial \omega}{\partial t} + \frac{U}{\varepsilon} \frac{\partial \omega}{\partial x} \right) = - \frac{\partial}{\partial z} \delta p - g \delta \rho + \Sigma_s T_s \left[\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \delta z_s \right] \\ \delta (z - z_s) - \frac{\mu}{k_1} \omega ,$$

$$(4) \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \omega}{\partial z} = 0 ,$$

$$(5) \quad \left(\varepsilon \frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right) \delta \rho = - \omega D \rho ,$$

$$(6) \quad \left(\varepsilon \frac{\partial}{\partial t} + U_s \frac{\partial}{\partial x} \right) \delta z_s = \omega_s ,$$

where $\delta (z - z_s)$ denotes Dirac's delta function, $D = d/dz$ and a subscript s distinguishes the value of the quantity at $z = z_s$.

Assuming the perturbation quantities to have x , y and t dependence of the form

$$(7) \quad \exp i (k_x x + k_y y + n t) ,$$

where k_x , k_y are horizontal wave numbers. $k^2 = k_x^2 + k_y^2$ and n is, in general, a complex constant.

Equations (1)-(6) give

$$(8) \quad D \left[\left\{ \frac{i \rho}{\varepsilon} \left(n + \frac{k_x U}{\varepsilon} \right) + \frac{\rho v}{k_1} \right\} D \omega - i k_x \omega \left(\frac{\rho}{\varepsilon^2} \right) D U \right] \\ - k^2 \left[\frac{i \rho}{\varepsilon} \left(n + \frac{k_x U}{\varepsilon} \right) + \frac{\rho v}{k_1} \right] \omega = i g k^2 (D \rho) \frac{\omega}{\varepsilon n + k_x U} , (z \neq z_s)$$

subjected to the boundary conditions :

$$(9) \quad \frac{\omega}{\varepsilon n + k_x U} \text{ must be continuous at an interface ,}$$

and

$$(10) \quad \Delta_s \left[\left\{ \frac{i\rho}{\varepsilon} \left(n + \frac{k_x U}{\varepsilon} \right) + \frac{\rho v}{k_1} \right\} D\omega + ik_x \left(\frac{\rho}{\varepsilon^2} \right) (DU)\omega \right] \\ = igk^2 \left[\Delta_s(\rho) - \frac{k^2 T_s}{g} \right] \left(\frac{\omega}{\varepsilon n + k_x U} \right)_s \quad (\text{for } z = z_s)$$

Consider now that two uniform fluids of densities ρ_1 and ρ_2 be separated by a horizontal boundary at $z=0$ and the density ρ_2 of the upper fluid be more than the density ρ_1 of the lower fluid so that, in the absence of streaming, the configuration is unstable and the medium throughout is assumed to be porous. Let the two fluids be streaming with constant velocities U_1 and U_2 and that the upper fluid layer is finite in width whereas the lower one is infinitely extending. Then in each region of constant ρ and U , Eq. (8) reduces to

$$(11) \quad (D^2 - k^2) \omega = 0.$$

Since $\omega_1 = 0$ as $z \rightarrow -\infty$ and $\omega_2 = 0$ at $z = d$ and since $\omega / (\varepsilon n + k_x U)$ must be continuous on the surface $z=0$, the appropriate solutions for the two regions are

$$(12) \quad \omega_1 = A_1 (\varepsilon n + k_x U_1) e^{+kz}, \quad (z < 0)$$

and

$$(13) \quad \omega_2 = A_2 (\varepsilon n + k_x U_2) e^{+kz} + B_2 (\varepsilon n + k_x U_2) e^{-kz}, \quad (z > 0)$$

where A_1 , A_2 and B_2 are arbitrary constants.

The condition $\omega_2 = 0$ at $z = d$ implies

$$(14) \quad B_2 = -A_2 e^{2kd},$$

and

$$\omega_2 = A_2 (\varepsilon n + k_x U_2) [e^{kz} - e^{(2d-z)k}].$$

The continuity of $\omega / (\varepsilon u + k_x U)$ on the surface $z=0$ gives

$$(15) \quad A_2 = A_1 / (1 - e^{2kd}),$$

and

$$(16) \quad \omega_2 = A_1 (\varepsilon n + k_x U_2) [e^{kz} - e^{(2d-z)k}] / (1 - e^{2kd}).$$

Applying the boundary condition (10) to the solutions (12) and (16), we obtain the dispersion relation

$$(17) \quad \{a_2 f(kd) - \alpha_1\} n^2 + \left[\frac{2k_x}{\varepsilon} \{a_2 U_2 f(kd) - \alpha_1 U_1\} - \frac{i\varepsilon}{k_1} \{a_2 v_2 f(kd) - \alpha_1 v_1\} \right] n \\ + \left[\frac{k_x^2}{\varepsilon_2} \left\{ a_2 U_2^2 f(kd) - \alpha_1 U_1^2 \right\} - \frac{ik_x}{k_1} \left\{ a_2 v_2 U_2 f(kd) - \alpha_1 v_1 U_1 \right\} - gk \left\{ \alpha_2 - \alpha_1 - \frac{k^2 T}{g(\rho_2 + \rho_1)} \right\} \right] = 0,$$

$$(18) \quad \text{where } f(kd) = \frac{1 + e^{2kd}}{1 - e^{2kd}}.$$

Equation (17) yields

$$(19) \quad Re(in) = \frac{-a \pm [(p^2 + Q^2)^{1/4} \cos \{ \frac{1}{2} \tan^{-1} (Q/p) \}]}{2 \{ \alpha_2 f(kd) - \alpha_1 \}}$$

where

$$(20) \quad a = \frac{\varepsilon}{k_1} \{ \alpha_2 v_2 f(kd) - \alpha_1 v_1 \},$$

$$P = \frac{\varepsilon^2}{k_1^2} \left\{ \alpha_2 v_2 f(kd) - \alpha_1 v_1 \right\}^2 - \frac{4k_x^2}{\varepsilon^2} \left\{ \alpha_2 U_2 f(kd) - \alpha_1 U_1 \right\}^2$$

$$+ 4 \left\{ \alpha_2 f(kd) - \alpha_1 \right\} \left[\frac{k_x^2}{\varepsilon^2} \left\{ \alpha_2 U_2^2 f(kd) - \alpha_1 U_1^2 \right\} \right. \\ \left. - gk \left\{ \alpha_2 - \alpha_1 - \frac{k^2 \Gamma}{g(\rho_1 + \rho_2)} \right\} \right],$$

$$Q = \frac{4k_x \alpha_1 \alpha_2 f(kd)}{k_1} (v_1 - v_2) (U_1 - U_2),$$

and

$$\alpha_{1,2} = \frac{\rho_{1,2}}{\rho_1 + \rho_2}, \quad v_{1,2} = \frac{\mu_{1,2}}{\rho_{1,2}}.$$

It is evident from Eq. (19) that, a necessary and sufficient criterion for stability of a more dense fluid layer of finite thickness flowing over a less dense fluid of infinite extent with a relative horizontal velocity in a porous medium is

$$(21) \quad (P^2 + Q^2) \cos^4 \left(\frac{1}{2} \tan^{-1} Q/P \right) < a^4,$$

where P , Q and are defined in Eqs. (20).

For the more dense fluid to be also infinitely extending ($f(kd) \rightarrow -1$) and for the non-porous medium ($\epsilon \rightarrow 1, k_1 \rightarrow \infty$), the criterion (21) reduces to that of Chandrasekhar (inequality (40), [1]). Also, for the case of more dense fluid of infinite extent and in the absence of streaming velocities (*i. e.* the case of Rayleigh - Taylor configuration), the criterion for stability (21) reduces to the result that the system is stable for all wave numbers

$k > \left(\frac{g(\rho_2 - \rho_1)}{T} \right)^{1/2}$ meaning thereby that the small wavelength perturbations are stabilized by surface tension.

Schubert and Straus [3] have shown that a more dense fluid phase overlying a less dense phase of the same fluid in a porous medium can be stable provided the permeability in the vicinity of the interface is not too large. This stabilization occurs due to the phase change and low permeability near the interface. In the present paper we have discussed the stability of a more dense fluid layer of finite thickness flowing over a less dense fluid of infinite extent with a relative horizontal velocity in porous medium. A necessary and sufficient criterion for stability has been derived.

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RAYLEIGH-TAYLOR INSTABILITY OF A FLUID-PARTICLE MIXTURE IN POROUS MEDIUM

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ABSTRACT

The Rayleigh - Taylor instability of a fluid - particle mixture in porous medium is considered . For the case of two uniform fluids separated by a horizontal boundary, the stable configuration (lighter fluid overlying the heavier one) remains stable but for unstable configuration (heavier fluid overlying the lighter one), the system is unstable and the growth rates may increase or decay with the increase in particle number density . For the case of exponentially varying density, there is instability for the unstable stratification and the growth rates are found to be both increasing or decreasing with the increase in suspended particle number density and medium permeability .

1. INTRODUCTION

The instability derived from the character of the equilibrium of an incompressible heavy fluid of variable density (*i. e.* of a heterogeneous fluid) is termed the Rayleigh-Taylor instability . Mention may

be made of two important special cases : (a) two fluids of different densities superposed one over the other ; (b) a fluid with a continuous density stratification . Chandrasekhar [1] has given a detailed account of the Rayleigh-Taylor instability in non-porous medium . When a fluid permeates a porous material, the gross effect is represented by the Darcy's law . As a result of this macroscopic law, the viscous form in the equations of fluid motion is replaced by the resistance term $(\mu/k_1) v$, where μ is the viscosity of the fluid, k_1 is the permeability of the medium and v is the Darcian (filter) velocity of the fluid . Wooding [4] has considered the Rayleigh instability of a thermal boundary layer in flow through a porous medium .

In geophysical situations . more often than not, the fluid is not pure but may instead be permeated with suspended (or dust) particles, Scanlon and Segel [2] have considered the effect of suspended particles on the onset of Bénard convection and found that the critical Rayleigh number was reduced solely because the heat capacity of the pure gas was supplemented by that of the particles , The thermal instability of fluids in a porous medium in the presence of suspended particles has been studied by Sharma and Sharma [3] . The effects of suspended particles and medium permeability were found to destabilize the layer .

Keeping in mind the geophysical situations and motivated by interest in fluid-particle mixtures, a study has been made of the Rayleigh-Taylor instability of a fluid-particle mixture in porous medium in the present note ,

2. Formulation of the Problem and Perturbation Equations

Consider an incompressible fluid-particle layer consisting of a fluid of density ρ , permeated with suspended particles of density mN ,

arranged in horizontal strata in porous medium. The system is acted on by gravity force $g(0, 0, -g)$. Let p , ρ and $v(u, v, w)$ denote respectively the pressure, density and velocity of the pure fluid; $u(1, r, s)$, m and $N(\bar{x}, t)$ denote the velocity, mass and number density of the particles respectively. $K = 6\pi\rho v\eta$, where η is particle radius, is constant, ϵ is porosity and $\bar{x} = (x, y, z)$. In the equations of motion for the fluid, the presence of particles adds an extra force term, proportional to the velocity difference between particles and fluid. The equations of motion and continuity for the fluid are

$$\frac{\rho}{\epsilon} \left[\frac{\partial v}{\partial t} + \frac{1}{\epsilon} (v \cdot \nabla) v \right] = -\nabla p + \rho g - \frac{\mu}{k_1} v + \frac{KN}{\epsilon} (u - v), \quad (1)$$

$$v \cdot v = 0, \quad (2)$$

Following a fluid particle in motion, we have

$$\epsilon \frac{\partial \rho}{\partial t} + (v \cdot \nabla) \rho = 0. \quad (3)$$

Since the force exerted by the fluid on the particles is equal and opposite to that exerted by the particles on the fluid, there must be an extra force term, equal in magnitude but opposite in sign, in the equations of motion for the particles. The buoyancy force on the particles is neglected. The interparticle reactions are also not considered because we assume that the distances between particles are quite large compared with their diameter. Under the above assumptions, the equations of motion and continuity for the particles are

$$mN \left[\frac{\partial u}{\partial t} + \frac{1}{\epsilon} (u \cdot \nabla) u \right] = KN (v - u), \quad (4)$$

$$\epsilon \frac{\partial N}{\partial t} + \nabla \cdot (N \mathbf{u}) = 0. \quad (5)$$

The initial state of the system is taken to be a quiescent layer (no settling) with a uniform particle distribution N_0 i. e. $\mathbf{v} = (0, 0, 0)$, $\mathbf{u} = (0, 0, 0)$ and $N = N_0$, is a constant, The character of the equilibrium of this initial static state can be determined, as usual, by supposing that the system is slightly disturbed and then by following its further evolution.

Let $\delta\rho$, δp , $\mathbf{v}(u, v, w)$ and $\mathbf{u}(l, r, s)$ denote respectively the perturbations in density ρ , pressure p , velocity of fluid and velocity of particles. Then the linearized perturbed forms of eqs. (1) - (5) become

$$\frac{\rho}{\epsilon} \frac{\partial \mathbf{v}}{\partial t} = -\nabla \delta p + g \delta \rho - \frac{\rho v}{k_1} \mathbf{v} + \frac{KN}{\epsilon} (\mathbf{v} - \mathbf{u}), \quad (6)$$

$$\nabla \cdot \mathbf{v} = 0,$$

$$mN \frac{\partial \mathbf{u}}{\partial t} = KN (\mathbf{v} - \mathbf{u}), \quad (8)$$

$$\epsilon \frac{\partial}{\partial t} = -\omega \frac{d\rho}{dz}, \quad (9)$$

where $\nu (= \mu/\rho)$ stands for kinematic viscosity of the fluid.

Analyzing the disturbances into normal modes, we seek solutions whose dependence on x , y and t is of the form

$$\exp(ik_x x + ik_y y + nt), \quad (10)$$

where n is a complex constant and k_x, k_y ($k^2 = k_x^2 + k_y^2$) are wave numbers along x and y directions respectively.

Eliminating u between eqs. (6) and (8) and using (10), eqs. (6)-(9) give

$$\left(n' + \frac{\nu}{k_1}\right) \rho u = -ik_x \delta p, \quad (11)$$

$$\left(n' + \frac{\nu}{k_1}\right) \rho v = -ik_y \delta p, \quad (12)$$

$$\left(n' + \frac{\nu}{k_1}\right) \rho w = -D\delta p - g\delta\rho, \quad (13)$$

$$ik_x u + ik_y v + D\omega = 0, \quad (14)$$

$$\epsilon n \delta\rho = -\omega D\rho, \quad (15)$$

where

$$n' = \frac{n}{\epsilon} \left(1 + \frac{uNK|\rho}{mu + K} \right) \text{ and } D = d/dz.$$

Eliminating δp between eqs. (11)-(13) and using eqs. (14) and (15), we obtain

$$\begin{aligned} & n' [D (\rho D\omega) - k^2 \rho \omega] + 1/k_1 [D (\rho v D\omega) - k^2 \rho v \omega] \\ & = - \frac{gk^2}{\epsilon n} (D\rho) \omega. \end{aligned} \quad (16)$$

3. Two uniform fluids separated by a horizontal boundary

Here we consider the case when two superposed fluids of uniform densities ρ_1 and ρ_2 and uniform viscosities μ_1 and μ_2 are separated by a horizontal boundary at $z=0$. The subscripts 1 and 2 distinguish the lower and the upper fluids respectively. Then, in each region of constant ρ and constant μ , eq. (16) reduces to

$$(D^2 - k^2) w = 0. \quad (17)$$

The general solution of eq. (17) is

$$w = A e^{+kz} + B e^{-kz}, \quad (18)$$

where A and B are arbitrary constants.

The boundary conditions to be satisfied in the present problem are :

- (i) The velocity w should vanish when $z \rightarrow +\infty$ (for the upper fluid) and $z \rightarrow -\infty$ (for the lower fluid).
- (ii) $w(z)$ is continuous at $z=0$.
- (iii) The pressure should be continuous across the interface.

Applying the boundary conditions (i) and (ii), we have

$$w_1 = A e^{+kz} \quad (z < 0), \quad (19)$$

$$w_2 = A e^{-kz} \quad (z > 0), \quad (20)$$

the same constant A has been chosen to ensure the continuity of w at $z = 0$.

The continuity of pressure means that

$$n' \Delta_0 \left(\rho D \omega \right) + \frac{1}{k_1} \Delta_0 \left(\rho v D \omega \right) + \frac{gk^2}{n\varepsilon} \Delta_0 (\rho) \omega_0 = 0. \quad (21)$$

Applying the condition (21) to the solutions (19) and (20), we obtain

$$\begin{aligned} n^3 + \left[\frac{K}{m} + \frac{2KN}{\rho_1 + \rho_2} + \frac{\varepsilon}{k_1} \left(\alpha_1 v_1 + \alpha_2 v_2 \right) \right] n^2 \\ + \left[\frac{K}{m} \frac{\varepsilon}{k_2} \left(\alpha_1 v_1 + \alpha_2 v_2 \right) + gk \left(\alpha_1 - \alpha_2 \right) \right] n + \frac{gkK}{m} \left(\alpha_1 - \alpha_2 \right) = 0, \end{aligned} \quad (22)$$

where $\alpha_{1,2} = \frac{\rho_{1,2}}{\rho_1 + \rho_2}$ and $v_{1,2} = \frac{\mu_{1,2}}{\rho_{1,2}}$.

(a) *Stable Case* ($\rho_1 > \rho_2$)

For potentially stable case, $\alpha_1 > \alpha_2$ and eq. (22) does not involve any change of sign and so does not allow any positive root. Therefore the system is stable.

(b) *Unstable Case* ($\rho_2 > \rho_1$)

For potentially unstable case, eq. (22) has one positive root and so the system is unstable. Let n_0 denote the positive root of eq. (22). Then eq. (22) yields

$$\begin{aligned} n_0^3 + \left[\frac{K}{m} + \frac{2KN}{\rho_1 + \rho_2} + \frac{\varepsilon}{k_1} \left(\alpha_1 v_1 + \alpha_2 v_2 \right) \right] n_0^2 \\ + \left[\frac{K}{m} \frac{\varepsilon}{k_2} \left(\alpha_1 v_1 + \alpha_2 v_2 \right) + gk \left(\alpha_1 - \alpha_2 \right) \right] n_0 + \frac{gkK}{m} \left(\alpha_1 - \alpha_2 \right) = 0. \end{aligned} \quad (23)$$

The presence of suspended particles, thus, does not affect the Rayleigh-Taylor instability in porous medium qualitatively. The denominator on the right-hand side of eq. (23) may be positive or negative. Thus with the increase in particle number density for the unstable case, the growth rates may increase or decay.

4. The Case of Exponentially Varying Density

Here we consider the density stratification in the fluid of depth d as

$$\rho(z) = \rho_0 e^{\beta z}, \quad (24)$$

where ρ_0 and β are constants. Assume that $[\beta d \ll 1, \text{i.e., the variation of density at two neighbouring points in the velocity field, which is much less than the average density, has a negligible effect on the inertia of the fluid,}$

The boundary conditions for the case of two free surfaces are

$$\omega = D^2\omega = 0 \text{ at } z = 0 \text{ and } z = d. \quad (25)$$

The proper solution of eq. (22) satisfying (25) is

$$\omega = W_0 \sin \frac{s\pi z}{d}, \quad (26)$$

where W_0 is a constant and s is an integer.

Substituting (26) in eq. (16) and neglecting the effect of heterogeneity of the inertia, we get

$$n^3 + \left[\frac{k}{m} \left(1 + \frac{mN}{\beta} \right) + \frac{\epsilon v}{k_1} \right] n^2 + \left[\frac{\epsilon v}{k_1} \frac{K}{m} - \frac{g\beta k^2}{\left(\frac{s\pi}{d} \right)^2 + k^2} \right] n$$

$$-\frac{g\beta k^2 K/m}{(\frac{s\pi}{d})^2 + k^2} = 0. \quad (27)$$

For the stable stratification ($\beta < 0$), eq. (27) does not have any positive root implying thereby that the system is stable. For the unstable stratification ($\beta > 0$), the constant term in eq. (27) is negative. This means that eq. (27) possesses one positive root implying thereby that the system is unstable. Let n_0 denote the positive root of eq (27). Then

$$n_0^3 + \left[\frac{K}{m} \left(1 + \frac{mN}{\rho} \right) + \frac{\epsilon v}{k_1} \right] n_0^2 + \left[\frac{\epsilon v}{k_1} \frac{K}{m} - \frac{g\beta k^2}{(\frac{s\pi}{d})^2 + k^2} \right] n_0 - \frac{g\beta k^2 K/m}{(\frac{s\pi}{d})^2 + k^2} = 0, \quad (28)$$

To find the role of particle number density and medium permability concerning the growth rate of unstable modes, we examine the natures of dn_0/dN and dn_0/dk_1 analytically. Equation (28) yields

$$\frac{dn_0}{dN} = \frac{(K/\rho) n_0^2}{\frac{g\beta k^2}{(\frac{s\pi}{d})^2 + k^2} - \left[3n_0^2 + 2n_0 \left\{ \frac{K}{m} \left(1 + \frac{mN}{\rho} \right) + \frac{\epsilon v}{k_1} \right\} + \frac{\epsilon v}{k_1} \frac{K}{m} \right]}, \quad (29)$$

and

$$\frac{dn_0}{dk_1} = \frac{\frac{\varepsilon v}{k_1^2} n_0 \left(n_0 + \frac{K}{m} \right)}{\left[3n_0^2 + 2n_0 \left\{ \frac{K}{m} \left(1 + \frac{\kappa N}{\rho} \right) + \frac{v}{k_1} \right\} + \frac{\varepsilon v}{k_1} \frac{K}{m} \right] - \left(\frac{s\pi}{d} \right)^2 + k^2} \quad (30)$$

The denominator on the right-hand side of eq. (29) may be positive or negative. Thus with the increase in particle number density for the unstable stratification, the growth rates may increase or decay. Similarly eq. (30) implies that with the increase in medium permeability, the growth rates may increase or decay for the unstable stratification.

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ON CERTAIN TRIPLE INTEGRAL RELATIONS

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ABSTRACT

The aim of this paper is to establish three triple integral relations involving elementary functions . A number of triple integrals can be deduced by proper specialization of the unknown functions g and f occurring in these relations . For the sake of illustration, one of our integral relations is applied to evaluate a general triple integral involving Srivastava and Panda's multivariable H -function .

1. INTRODUCTION

Many authors have worked on the problem of obtaining integral relations involving higher classes of special functions of one and more variables (see [3, pp. 72-74 ; pp . 156-161] for details). In this paper we derive three new integral relations associated with some elementary functions and illustrate how they can be applied to derive triple integrals which may be of interest .

2. INTEGRAL RELATIONS

$$\int_0^\infty \int_0^\infty \int_0^\infty (x^2 + y^2)^{-1/2} \exp [i (x^2 + y^2 + z^2) (\frac{x^2 - y^2}{x^2 + y^2})]$$

$$\cos [2n \tan^{-1} (y/x)] f (x^2 + y^2 + z^2)$$

$$\cdot g \left\{ \tan^{-1} \left(\frac{x^2+y^2}{2} \right) \right\} dx dy dz$$

$$= \frac{\pi i^n}{2} \int_0^\infty \int_0^\infty J_n(u^2+v^2) f(u^2+v^2) g \left\{ \tan^{-1} \left(\frac{v}{u} \right) \right\} du dv, \quad \dots (2.1)$$

where n is any integer, positive or negative, and the functions f and g are so constrained that the various integrals involved in (2.1) exist.

$$\int_0^\infty \int_0^\infty \int_0^\infty (xy)^{\nu+1/2} (x^2+y^2)^{-(\nu+1)} \exp \left\{ i(x^2+y^2+z^2) \left(\frac{x^2-y^2}{x^2+y^2} \right) \cos \Psi \right\}$$

$$\cdot J_{\nu-1/2} \left[\frac{2xy}{(x^2+y^2)} (x^2+y^2+z^2) \sin \Psi \right] C_n^\nu \left(\frac{x^2-y^2}{x^2+y^2} \right) f(x^2+y^2+z^2)$$

$$g \left[\tan^{-1} \left(\frac{x^2+y^2}{z} \right) \right] dx dy dz$$

$$= 2^{-(\nu+1)} \sqrt{\pi i^n} (\sin \Psi)^{\nu-1/2} C_n^\nu (\cos \Psi)$$

$$\cdot \int_0^\infty \int_0^\infty (u^2+v^2)^{-1/2} J_{\nu+1/2}(u^2+v^2) f(u^2+v^2) g \left[\tan^{-1} v/u \right] du dv \quad \dots (2.2)$$

provided that $\text{Re}(\nu) > -1/2$, $n = 0, 1, 2, \dots$, and f and g are so constrained that the various integrals involved in (2.2) exist.

$$\int_0^\infty \int_0^\infty \int_0^\infty (xy)^{2\nu} (x^2+y^2)^{-(2\nu+1/2)} W^{-\nu} Z_\nu(W)$$

$$C_n^v \left(\frac{x^2 - y^2}{x^2 + y^2} \right) f(x^2 + y^2 + z^2) g \left[\tan^{-1} \left(\frac{x^2 + y^2}{z} \right) \right] dx dy dz$$

$$= \frac{2^{3v} \pi \Gamma(n + 2v) Z_{v+n}(t) t^{-v}}{n! \Gamma(v)}$$

$$\int_0^\infty \int_0^\infty (u^2 + v^2)^{-v} J_{v+n}(u^2 + v^2) f(u^2 + v^2) g \left[\tan^{-1} v/u \right] du dv \dots (2.3)$$

valid under the same conditions as those stated for (2.2) above.

In (2.1), (2.2) and (2.3) $J_v(x)$ is the Bessel function of the first kind, $C_n^v(x)$ is the Gegenbauer polynomial, and $Z_v(x)$ stands for any Bessel function of the first, second, or third kind. Also

$$W = [(x^2 + y^2 + z^2)^2 + t^2 - 2t \left(\frac{x^2 - y^2}{x^2 + y^2} \right) (x^2 + y^2 + z^2)]^{-1/2} \dots (2.4)$$

Proof of (2.1): We have [5, p. 409, Ex. 3]

$$J_n(z) = \frac{1}{\pi i^n} \int_0^\pi \exp(iz \cos \theta) \cos n\theta d\theta \dots (2.5)$$

In order to derive the integral relation (2.1), we replace z by r^2, θ by 2θ in (2.5), multiply both sides by $rf(r^2)g(\phi)drd\phi$, and then integrate the resulting equation with respect to r and ϕ over the intervals $(0, \infty)$ and $(0, \pi/2)$, respectively. We thus get

$$\frac{\pi i^n}{2} \int_0^\infty \int_0^{\pi/2} \{J_n(r^2) r f(r^2) g(\phi) dr\} d\phi$$

$$= \int_0^\infty \int_0^{\pi/2} \int_0^{\pi/2} \exp(ir^2 \cos 2\theta) \cos 2n\theta \frac{r \sin \phi}{r \sin \phi} r f(r^2) g(\phi) dr d\phi d\theta \quad \dots (2.6)$$

If we make the substitutions $x=r \sin \phi \cos \theta$, $y=r \sin \phi \sin \theta$, and $z=r \cos \phi$ on right-hand side of (2.6), and set $u=r \cos \phi$, $v=r \sin \phi$ in left hand side, we are easily led to the integral relation (2.1).

To prove the integral relations (2.3) and (2.4), we start with the known integrals [1, p. 57, Eq. (12); p. 48, Eq. (14)] and proceed on the lines similar to those mentioned in the proof of (2.1).

3. USEFUL DEDUCTIONS

The function g appearing in our integrals (2.1) (2.2) and (2.3) may be chosen appropriately to derive various triple integrals.

For example, if in (2.1), we get

$$g(t) = \cos 2\mu t (\sin t)^v \quad \dots (3.1)$$

and simplify the right-hand side of the resulting equation by means of a known integral relation [3, p. 72, Eq. (5.4.3)], we arrive at the following result :

$$\int_0^\infty \int_0^\infty \int_0^\infty (x^2+y^2)^{(v-1)/2} (x^2+y^2+z^2)^{-v/2} \exp \{ i (x^2+y^2+z^2) \}$$

$$\left(\frac{x^2-y^2}{x^2+y^2} \right) \cos [2n (\tan^{-1} y/x)] \cos [2\mu \{ \tan^{-1} \left(\frac{x^2+y^2}{z} \right) \}]$$

$$f(x^2+y^2+z^2) dx dy dz$$

$$= \frac{\sqrt{\pi} i^n \Gamma(\frac{1}{2} \pm \mu) \Gamma(\frac{1+\nu}{2}) \Gamma(1+\nu/2)}{8 \Gamma(1+\nu/2 \pm \mu)} \int_0^\infty f(t) J_n(t) dt \quad \dots (3.2)$$

$\text{Re}(\nu) > 0$, n is an integer, positive or negative, and f is so chosen that the integrals on both sides of (3.2) exist.

Now in (3.2), we set

$$f(t) = t^{\lambda-1} H[z_1 t^{\rho_1}, \dots, z_r t^{\rho_r}]$$

where

$$H[z_1, \dots, z_r] \equiv H \begin{matrix} O, O: M_1, N_1; \dots; M_r, N_r \\ P, Q: P_1, Q_1; \dots; P_r, Q_r \end{matrix}$$

$$\begin{bmatrix} z_1 \\ \vdots \\ z_r \end{bmatrix} \left(\begin{matrix} a_j; \alpha'_j, \dots, \alpha^{(r)}_j \\ b_j; \beta'_j, \dots, \beta^{(r)}_j \end{matrix} \right)_{1, P: Q}$$

$$\left(\begin{matrix} c'_j, \gamma'_j \\ d'_j, \delta'_j \end{matrix} \right)_{1, P_1}; \dots; \left(\begin{matrix} c^{(r)}_j, \gamma^{(r)}_j \\ d^{(r)}_j, \delta^{(r)}_j \end{matrix} \right)_{1, P_r} \left(\begin{matrix} c'_j, \gamma'_j \\ d'_j, \delta'_j \end{matrix} \right)_{1, Q_1}; \dots; \left(\begin{matrix} c^{(r)}_j, \gamma^{(r)}_j \\ d^{(r)}_j, \delta^{(r)}_j \end{matrix} \right)_{1, Q_r}$$

is a special case of the multivariable H -function due to Srivastava and Panda [4] (see also [3, p. 251, Eq. (C.1)]).

Evaluating the resulting integral with the help of a known integral [2, p. 272, Eq. (3.6)], we arrive at the following interesting triple integral which is believed to be new:

$$\int_0^\infty \int_0^\infty \int_0^\infty (x^2+y^2+z^2)^{\lambda-(\nu+2)/2} (x^2+y^2)^{(\nu-1)/2} \exp [i(x^2+y^2+z^2)]$$

$$\left(\frac{x^2-y^2}{x^2+y^2} \right) \cos [2n (\tan^{-1} y/x)] \cos \left\{ 2\mu (\tan^{-1} \frac{x^2+y^2}{z}) \right\}$$

$$H [Z_1 (x^2+y^2+z^2)^{\rho_1}, \dots, Z_r (x^2+y^2+z^2)^{\rho_r}] dx dy dz$$

$$2^{\lambda-4} \sqrt{\pi} i^n \Gamma \left(\frac{1}{2} \pm \mu \Gamma \left(\frac{1+\nu}{2} \right) \Gamma \left(1 + \frac{\nu}{2} \right) \right)$$

$$= \frac{\Gamma \left(+ \frac{\nu}{2} \pm \mu \right)}{\Gamma \left(+ \frac{\nu}{2} \pm \mu \right)}$$

$$H \begin{matrix} O, 1 : M_1, N_1 ; & , & M_r, N_r \\ P+2, Q : P_1, Q_1 ; & \dots ; & P_r, Q_r \end{matrix} \left[\begin{matrix} Z_1 2^{\rho_1} \\ \vdots \\ Z_r 2^{\rho_r} \end{matrix} \right] \left(1 \mp \frac{n}{2} - \frac{\lambda}{2} ; \frac{\rho_1}{2}, \dots, \frac{\rho_r}{2} \right)$$

$$\left[\begin{matrix} (a_j ; \alpha_j', \dots, \alpha_j^{(r)})_{1,P} ; (c_j', \gamma_j')_{1,P_1} ; \dots ; (c_j^{(r)}, \gamma_j^{(r)})_{1,P_r} \\ (b_j ; \beta_j', \dots, \beta_j^{(r)})_{1,Q} ; (d_j', \delta_j')_{1,Q_1} ; \dots ; (d_j^{(r)}, \delta_j^{(r)})_{1,Q_r} \end{matrix} \right]$$

... (3.3)

provided that $\operatorname{Re}(\nu) > 0$, $\rho_i > 0$ ($i = 1, \dots, r$), n is an integer, positive or negative,

$$\operatorname{Re}(\lambda) + \sum_{i=1}^r [\rho_i \min_{1 \leq j \leq m_i} \operatorname{Re}(d_j^{(i)} / \delta_j^{(i)})] + n > 0,$$

$$\operatorname{Re}(\lambda) + \sum_{i=1}^r \left[\rho_i \max_{1 \leq j \leq n_i} \operatorname{Re} \left(\frac{c_j^{(i)} - 1}{\gamma_j^{(i)}} \right) \right] - \frac{3}{2} < 0$$

$$\Lambda_i = - \sum_{j=1}^P \alpha_j^{(i)} - \sum_{j=1}^Q \beta_j^{(i)} + \sum_{j=1}^{N_i} \gamma_j^{(i)} - \sum_{j=n_i+1}^{P_i} \gamma_j^{(i)}$$

$$+ \sum_{j=1}^{M_i} \delta_j^{(i)} - \sum_{j=m_i+1}^{Q_i} \delta_j^{(i)} > 0$$

and

$$|\arg(Z_i)| < \frac{1}{2} \Lambda_i \pi, \quad (i = 1, \dots, r).$$

The triple integral (3.3) is quite general in character due to general nature of the multivariable H -function involved therein. Thus, by appropriately reducing this multivariable H -function in terms of simpler special functions, one can easily obtain a considerably large number of triple integrals of interest to mathematical analysts and applied mathematicians. The details are omitted.

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GENERALIZED FIXED POINT THEOREMS IN METRIC SPACE

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ABSTRACT

B. E. Rhoades proved several fixed point theorems for mapping satisfying much more general contractive type conditions for three points than those of pittnauer (1975). They have taken only single operator. In this paper we proved fixed point theorems involving three and four points and at the same time two and more operators. These theorems are more general than those of pittnauer and Rhoades

Theorem 1.

Let T_1 and T_2 be two operators mapping on a complete metric space X , with metric d into itself such that $T_1^p T_2^q(x) = T_2^q T_1^p(x)$, for $p, q \geq 1$ and $T_1^p T_2^q(x)$ means T_1 operating p times and T_2 operating q times. Further if for a real number h satisfying $0 \leq h \leq 1$ and for each $x, y, z \in X$,

$$d(T_1(x), T_2(y)) \leq h \max \{d(x, y), d(x, z), d(y, z)\}.$$

$$d(x, T_1(x)), d(y, T_2(y)), d(x, T_2(y)),$$

$$d(y, T_1(x)), [d(x, T_1^p T_2^q(z)) + d(y, T_1^p T_2^q(z))],$$

$$\frac{[d(x, T_1^p T_2^q(z)) + d(T_1(x), T_1^p T_2^q(z))]}{2},$$

$$\frac{[d(y, T_1^p T_2^q(z)) + d(T_2(y), T_1^p T_2^q(z))]}{2},$$

$$\frac{[d(x, T_1^p T_2^q(z)) + d(T_2(y), T_1^p T_2^q(z))]}{2},$$

$$\frac{[d(y, T_1^p T_2^q(z)) + d(T_1(x), T_1^p T_2^q(z))]}{2}, \quad (1)$$

Then T_1, T_2 have a unique common fixed point .

Proof .

Let $x_0 \in X$ and consider a sequence $\{x_n\}$ by setting

$$x_1 = T_1(x_0), x_2 = T_2(x_1), x_3 = T_1(x_2), \dots$$

For any set A , let $\delta(A)$ denotes the diameter of A .

Assume, $O(x_0, n) = \{x_0, x_1, \dots, x_n\}$

and $O(x_0, \infty) = \{x_0, x_1, \dots\}$

Let $K \leq i < j < n, x = x_i, y = x_j$

where i is an even integer and j is an odd integer,

$$z = x_{i-k} \text{ and } p+q=k.$$

Now,

$$\begin{aligned} d(x_{i+1}, x_{j+1}) &= d(T_1(x_i), T_2(x_j)) \\ &\leq h \max \{ d(x_i, x_j), d(x_i, x_{i-k}), d(x_j, x_{j-k}), \\ &\quad d(x_i, x_{i+1}), d(x_j, x_{j+1}), d(x_i, x_{j+1}), d(x_j, x_{i+1}) \} \\ &\leq h \delta [0(x_{i-k}, j+1+k-i)] \\ &\leq h \delta [0(x_{i-k}, n)] \dots \end{aligned} \tag{2}$$

Note that $\delta [0(x_{i-k}, 1)] \leq \delta [0(x_{i-k}, 2)] \leq \dots$

so that $\delta [0(x_{i-k}, \infty)] = \sup_n \delta [0(x_{i-k}, n)]$.

From (2) it follows that,

$$\delta [0(x_{i-k}, n)] = d(x_c, x_t)$$

for some choice of integer c, t satisfying,

$$i-k \leq c \leq t, c < t \leq n.$$

Let n be any positive integer. Then there exists integers c and t with

$$i-k \leq c \leq i, c < t \leq n \text{ such that}$$

$$\delta [0(x_{i-k}, n)] = d(x_c, x_t)$$

If $t \leq i+1$, then

$$d(x_c, x_t) \leq M = \max \{ d(x_c, x_u) : i-k \leq c < u \leq i+1 \}$$

If $t > i$, then,

$$d(x_c, x_t) \leq d(x_c, x_{i+1}) + d(x_{i+1}, x_t)$$

$$\leq M + h \delta [0(x_{i-k}, n)]$$

$$\leq M + h d(x_c, x_t) \text{ and } d(x_c, x_t) \leq \frac{M}{1-h}$$

Therefore, $\delta [0(x_{i-k}, \infty)] \leq \frac{M}{1-h}$.

Let $m > n > k$, From (2) with

$$i+1=n, j+1=m,$$

$$d(x_m, x_n) \leq h \delta [0(x_{n-k-1}, m+k-n+1)]$$

Then there exists a choice of integers c_1, t_1 satisfying $0 \leq c_1 \leq k$, $c_1 < t_1 \leq m+k-n+1$ such that

$$\begin{aligned} \delta [0(x_{n-k-1}, m+k-n+1)] &= d(x_{n-k-1+c_1}, x_{n-k-1+t_1}) \\ &= d[T_1(x_{n-k-2+c_1}), T_2(x_{n-k-2+t_1})] \\ &\leq h \delta [0(x_{n-2k-2+c_1}, t+k+1-c_1)] \\ &\leq h \delta [0(x_{n-2k-2}, m-n+2k+2)]. \end{aligned}$$

So that,

$$d(x_m, x_n) \leq h^2 \delta [0(x_{n-2k-2}, m-n+2k+2)]$$

In general, with r satisfying

$$n = r(k+1) + s, 0 \leq s < k+1.$$

$$d(x_m, x_n) \leq h^r \delta [0(x_s, m-n+r(k+1))]$$

$$\leq h^r \delta [0(x_0, m)]$$

$$\leq h^r \frac{M}{1-h} \rightarrow 0 \text{ as } r \rightarrow \infty.$$

Therefore $\{x_n\}$ is a Cauchy sequence, hence convergent to a limit say x' .

Now for even n , consider $x=x_n, y=x', z=x_{n-k}, p+q=k$.

Then by (1) we have,

$$d(x_{n+1}, T_2(x')) = d(T_1(x_n), T_2(x'))$$

$$\leq h \max \{d(x_n, x'), d(x_n, x_{n-k}),$$

$$d(x', x_{n-k}), d(x_n, x_{n+1}), d(x', T_2(x')),$$

$$d(x_n, T_2(x')); d(x', x_{n+1})\}$$

Taking the limit as $n \rightarrow \infty$, we obtain

$$d(x', T_2(x')) \leq h d(x', T_2(x')) \Rightarrow T_2(x') = x'$$

i.e., x' is the fixed point of T_2 .

Similarly by taking n odd and

$$x=x', y=x_n, z=x_{n-k}$$

We get from (1),

$$\begin{aligned} d(T_1(x'), x_{n+1}) &= d(T_1(x'), T_2(x_n)) \\ &\leq h \{ d(x', x_n) d(x', x_{n-k}), d(x_n, x_{n-k}), \\ &\quad d(x', T_1(x')), d(x_n, x_{n+1}), d(x', x_{n+1}), \\ &\quad d(x_n, T_1(x')) \} . \end{aligned}$$

Then taking the limit as $n \rightarrow \infty$, we obtain

$$d(T_1(x'), x') \leq h d(x', T_1(x')) \Rightarrow T_1(x') = x' ,$$

Showing that x' is the fixed point of T_1 ,

Thus x' is a common fixed point of T_1 and T_2 .

Now we shall show that there is no other common fixed point y' of T_1 and T_2 . For if y' were such a point

$$\text{then, } T_1(y') = T_2(y') = y', T_1(x') = T_2(x') = x' .$$

considering $x = z = x', y = y'$, we get

$$d(x', y') = d(T_1(x'), T_2(y')) \leq h d(x', y')$$

showing that $x' = y'$

Remark .

If we put $T_1 = T_2$ in theorem 1, we get the following theorem of Rhoades :

Let f be a self mapping of a complete metric space X which satisfying for each $x, y, z \in X, 0 \leq h \leq 1$ and a positive integer k ,

$$\begin{aligned}
d(f(x), f(y)) &\leq h \max \{ d(x, y), d(x, z), d(y, z), d(x, f(x)), \\
&\quad d(y, f(y)), d(x, f(y)), d(y, f(x)), \\
&\quad [d(x, f^k(z)) + d(y, f^k(z))], [d(x, f^k(z)) + d(f(x), f^k(z))], \\
&\quad \frac{[d(y, f^k(z)) + d(f(y), f^k(z))], [d(x, f^k(z)) + d(f(y), f^k(z))]}{2}, \\
&\quad \frac{[d(y, f^k(z)) + d(f(x), f^k(z))]}{2} \}.
\end{aligned}$$

Theorem 2.

Consider for any positive integers k and m , the self mapping T of a complete metric space X satisfies for $x, y, z_1, z_2 \in X$, $x \neq y$,

$$\begin{aligned}
d(T(x), T(y)) &< \max \{ d(x, y), d(x, T(x)), d(y, T(y)), \\
&\quad \frac{[d(x, T(y)) + d(y, T(x))], [d(x, T^k(z_1)) + d(y, T^k(z_1))], \\
&\quad [d(x, T^m(z_2)) + d(y, T^m(z_2))], [d(y, T^k(z_1)) + d(T(y), T^k(z_1))]}{2}, \\
&\quad \frac{[d(y, T^m(z_2)) + d(T(y), T^m(z_2))], \\
&\quad [d(x, T^k(z_1)) + d(T(y), T^k(z_1))], \\
&\quad [d(x, T^m(z_2)) + d(T(y), T^m(z_2))], \\
&\quad [d(x, T^k(z_1)) + d(T(x), T^k(z_1))], \\
&\quad [d(x, T^m(z_2)) + d(T(x), T^m(z_2))]}{2} \dots
\end{aligned} \tag{1}$$

If $T^n(x_0)$ has a cluster point p for some $x_0 \in X$ and if T is continuous on $\text{cl } 0(x_0)$, the closure of the orbit of x_0 , then p is the unique fixed point of T and $T^n(x_0) \rightarrow p$.

Proof. Let $x_0 \in X$ and consider the sequence $\{x_n\}$,

$$\{x_0, x_1 = T(x_0), \dots, x_{n+1} = T(x_n), \dots\}.$$

For each $k, m \leq i$, take, $x = x_i, y = x_{i-1}, z_1 = x_{i-k}, z_2 = x_{i-m}$.

Then we have from (1),

$$d(x_{i+1}, x_i) \leq \max \{d(x_i, x_{i-1}), d(x_i, x_{i+1}), d(x_i, x_{i+1}), d(x_{i-1}, x_i),$$

$$\frac{[d(x_i, x_i) + d(x_{i-1}, x_{i+1})], [d(x_i, x_i) + d(x_{i-1}, x_i)]}{2},$$

$$[d(x_i, x_i) + d(x_{i-1}, x_i)], \frac{[d(x_{i-1}, x_i) + d(x_i, x_i)]}{2},$$

$$\frac{[d(x_{i-1}, x_i) + d(x_i, x_i)]}{2}, [d(x_i, x_i) + d(x_i, x_i)],$$

$$[d(x_i, x_i) + d(x_i, x_i)], [d(x_i, x_i) + d(x_{i+1}, x_i)],$$

$$[d(x_i, x_i) + d(x_{i+1}, x_i)]$$

$$i. e., d(x_{i+1}, x_i) \leq \max \{d(x_i, x_{i-1}), \frac{(x_{i-1}, x_{i+1})}{2}\} \dots (2)$$

We have for $x_{i+1} \neq x_i$, since each $i \geq k, m$; otherwise $\{T^n(x_0)\}$ automatic converges to a fixed point.

Suppose the maximum of the right hand side of (2) is

$$\frac{d(x_{i-1}, x_{i+1})}{2}.$$

Then we have ,

$$\begin{aligned} d(x_{i-1}, x_i) &< \frac{d(x_{i-1}, x_{i+1})}{2} \\ &\leq \left[\frac{d(x_{i-1}, x_i) + d(x_i, x_{i+1})}{2} \right] \end{aligned}$$

which implies $d(x_{i-1}, x_i) < d(x_i, x_{i+1})$... (3)

Also $d(x_{i+1}, x_i) < \frac{d(x_{i-1}, x_{i+1})}{2}$ which implies

$$d(x_{i+1}, x_i) < \frac{d(x_{i-1}, x_i)}{2} + \frac{d(x_i, x_{i+1})}{2}$$

i e., $d(x_{i+1}, x_i) < \frac{d(x_{i-1}, x_i)}{2} < d(x_{i-1}, x_i)$... (4)

Combining (3) and (4) we have the contradiction,

$$d(x_{i+1}, x_i) < d(x_{i+1}, x_i).$$

Therefore from (2), $d(x_{i+1}, x_i) < d(x_i, x_{i-1})$ [4, Lemma 2]

This shows that $T(p) = p$.

Suppose p and w are fixed points of T , $p \neq w$.

Then from (1) we obtain the contradiction, $d(p, w) < d(p, w)$.

Since $T^n(x_0) \rightarrow p$ for some subsequence $\{n_i\}$ and T is continuous on $cl\,0(x_0)$,

$$\begin{aligned} d(T^{n_i}(x_0), T^{n_{i+1}}(x_0)) &= d(x_{n_i}, x_{n_{i+1}}) \rightarrow d(p, T(p)) \\ &= 0 \Rightarrow T(p) = p. \end{aligned}$$

Fix $\varepsilon > 0$. Then there exists an integer j such that $i \geq j$ implies

$$d(x_{n_i}, x_{n_{i+1}}) < \varepsilon \text{ and } d(x_{n_i}, p) < \varepsilon$$

If we choose $x = x_{n-1}$, $y = p$, $z_1 = x_{n-k-1}$ and $z_2 = x_{n-m-1}$,

then for each $n > n_{j+k}$, $n > n_{j+m}$, we have from (1) and (4),

$$\begin{aligned} d(x_n, p) &< \max\{d(x_{n-1}, p), d(x_{n-1}, x)\} \\ &< \max\{d(x_{n-2}, p), d(x_{n-2}, x_{n-1}), d(x_{n-1}, x_n)\} \\ &= \max\{d(x_{n-2}, d(x_{n-2}, x_{n-1}))\} \text{ from (4)} \\ &< \dots\dots\dots \\ &< \max\{d(x_{n_j}, x_{n_{j+1}}), d(x_{n_j}, p)\} < \varepsilon \end{aligned}$$

i. e. $T^n(x_0) \rightarrow p$.

Theorem 3.

Let T_1 and T_2 be two mappings on a complete metric space X on itself such that $T_1^p T_2^q = T_2^q T_1^p$; $p, q \geq 1$, then for every $x, y, z, z_2 \in X$ and $0 < h < 1$,

$$d(T_1(x), T_2(y)) \leq h \max \{ d(x, y), d(x, T_1(x)), d(y, T_2(x)), \\ \frac{[d(x, T_2(y)) + d(y, T_1(x))]}{2} \},$$

$$[d(x, T_1^p T_2^q(z_1)) + d(y, T_1^p T_2^q(z_1))],$$

$$[d(x, T_2^p T_2^q(z_2)) + d(y, T_1^p T_2^q(z_2))],$$

$$[d(x, T_1^p T_2^q(z_1)) + d(T_1(x), T_1^p T_2^q(z_1))],$$

$$[d(x, T_1^p T_2^q(z_2)) + d(T_1(x), T_1^p T_2^q(z_2))],$$

$$[d(y, T_1^p T_2^q(z_1)) + d(T_2(y), T_1^p T_2^q(z_1))],$$

$$[d(y, T_1^p T_2^q(z_2)) + d(T_2(y), T_1^p T_2^q(z_2))];$$

$$[d(x, T_1^p T_2^q(z_1)) + d(T_2(y), T_1^p T_2^q(z_1))],$$

$$[d(x, T_1^p T_2^q(z_2)) + d(T_2(y), T_1^p T_2^q(z_2))] \quad \dots (1)$$

Then T_1 and T_2 have a unique common fixed point.

Proof .

Let $x_0 \in X$ and construct a sequence $\{x_n\}$ as follows :

$x_0, T_1(x_0) = x_1, T_2(x_1) = x_2, T_1(x_2) = x_3, T_2(x_3) = x_4$, and so on.

Consider $x = x_i$ (i - even),

$y = x_{i-1}, z_1 = x_{i-p-q}, z_2 = x_{i-p-q-1}$. Then from (1) we have,

$$d(T_1(x), T_2(y)) = d(T_1(x_i), T_2(x_{i-1})) = d(x_i, x_{i+1})$$

$$\leq h \max \{ d(x_i, x_{i-1}), d(x_i, x_{i+1}), d(x_{i-1}, x_i) \},$$

$$\left[\frac{d(x_i, x_i) + d(x_{i-1}, x_{i+1})}{2} \right], [d(x_i, x_i) + d(x_{i-1}, x_i)],$$

$$[d(x_i, x_i) + d(x_{i-1}, x_i)], [d(x_i, x_i) + d(x_{i-1}, x_i)],$$

$$[d(x_i, x_i) + d(x_{i-1}, x_i)], [d(x_{i-1}, x_i) + d(x_i, x_i)],$$

$$[d(x_{i-1}, x_i) + d(x_i, x_i)], [d(x_i, x_i) + d(x_i, x_i)],$$

$$[d(x_i, x_i) + d(x_i, x_i)] \},$$

$$\leq h \max \{ d(x_i, x_{i-1}), d(x_i, x_{i+1}), d\left(\frac{x_{i-1} + x_{i+1}}{2}\right) \}$$

$$< h \max \{ d(x_i, x_{i-1}), d\left(\frac{x_{i-1} + x_{i+1}}{2}\right) \}.$$

This implies that $d(x_i, x_{i+1}) \leq h d(x_i, x_{i-1})$

i. e., $d(T_1(x), T_2(y)) \leq h d(x, y)$.

This shows that T_1 and T_2 have a common fixed point [6].

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ON A BEST APPROXIMATION THEOREM

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Th

ABSTRACT

Defin.

In this paper a Theorem on Best Approximation is proved. It extends $G(x)$ to Ky Fan, Hadzic, and Sehgal, Singh and Smithson.

A well known theorem due to Ky Fan [2] has been of great importance in nonlinear analysis, minimax theory and approximation theory. This theorem has been further extended by several mathematicians and applied in various areas including game theory and the theory of Economics.

In this paper we state a best approximation theorem in a topological vector space which contains as particular cases the above, an other result recently given by Ky Fan [3], and its generalizations due

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to Sehgal, Singh and Smithson [10], and Hadzic [4]. This theorem is an useful tool to obtain nearest point theorems, coincidence theorems and fixed point theorems.

We need the following definition :

Let E be a vector space and p a seminorm on E , X a non-empty convex subset of E . A map $g : X \rightarrow E$ is said to be almost p -affine if for all $x, y \in X$ $u \in E$ and r , $0 < r < 1$ satisfies

$$(1) \quad p(g(rx + (1-r)y) - u) \leq rp(gx - u) + (1-r)p(gy - u).$$

Now we state our main theorem :

Theorem 1.

Let (E, T) be a topological vector space, X a non-empty subset of E , S a non-empty compact convex subset of X and K a compact subset of X . Let p be a seminorm on E , $f : (X, T) \rightarrow (Y, S)$ a continuous map and $g : X \rightarrow E$ be an almost p -affine map such that $p \circ g$ is lower semicontinuous on (X, T) . If f satisfies the condition :

$$(2) \quad \text{for every } y \text{ in } X/K \text{ there exists an } x \in S \text{ satisfying } p(gx - fy) < p(gy - fy),$$

then there exists a $u \in K$ such that

$$(3) \quad p(gu - fu) = \min \{ p(gx - fu) : x \in X \}.$$

If moreover, g is onto X , then we have

$$(4) \quad p(gu - fu) = \min \{ p(x - fu) : x \in X \}.$$

Proof

The proof is similar to that of Ky Fan's Theorem 7 in [3]. In the proof we need the following theorem due to Allen [1].

Theorem (A) :

Let X be a non-empty convex subset of a topological vector space E . Let $f: X \times X \rightarrow R$ be such that

- (i) for each fixed $x \in X$, $f(x, y)$ is a l. s. c. function of y on X .
- (ii) for each fixed $y \in X$, $f(x, y)$ is a quasi-concave function of x on X ,
- (iii) $f(x, x) \leq 0$ for all $x \in X$.

Furthermore,

- (iv) let X_0 be a compact, convex non-empty subset of X such that the set $\{y \in X : f(x, y) \leq 0 \text{ for all } x \in X_0\}$ is compact.

Then there exists a $y_0 \in X$ such that $f(x, y_0) \leq 0$ for all $x \in X$.

Define $G: X \times X \rightarrow R$ by

$$G(x, y) = p(gy - fy) - p(gx - fy).$$

For every $x \in X$, G is a lower semicontinuous function of y and $G(x, x) = 0$.

Now we show that $G(x, y)$ is a quasi-concave function of x for every $y \in X$.

Indeed, if $G(x_1, y) > t$ and $G(x_2, y) > t$, then because of (1) $G(rx_1 + (1-r)x_2, y) \geq rG(x_1, y) + (1-r)G(x_2, y) > t$.

Finally, we show that the set

$$A = \{y \in X : G(x, y) \leq 0 \text{ for all } x \in S\} \text{ is compact.}$$

Indeed, if $A = \phi$ then it is obvious. If $A \neq \phi$, then since G is a lower semicontinuous function of y for all $x \in S$, A is a closed set; because $y \in A$ if and only if $y \in X$ and $p(gy - fy) \leq p(gx - fy)$ for all $x \in S$, A is a subset of K by (2). Therefore A is compact.

Thus G satisfies all the conditions of Theorem (A), therefore there exists a $u \in X$ such that $G(x, u) \leq 0$ for all $x \in X$. Moreover, $u \in A \subset K$, hence $u \in K$ and satisfies (3).

If, in particular, X is a compact set, we may assume in Theorem 1, $X = S = K$. The Condition (2) is obviously satisfied and we obtain as a corollary the following :

Theorem 2.

Let (E, T) be a topological vector space, $K \subset E$ a non-empty convex compact set and p a seminorm on E . Let $f : (K, T) \rightarrow (E, p)$ be a continuous map and $g : K \rightarrow E$ be an almost p -affine map such that $p \circ g$ is lower semicontinuous on (K, T) . Then there exists a $y \in K$ such that $p(fy - gy) = \min \{p(fy - gx) : x \in K\}$.

If, moreover, g is onto K , we have

$$p(fy - gy) = \min \{p(fy - x) : x \in K\}.$$

Now we derive some consequences of Theorems 1 and 2 in order to obtain various results as corollaries in Approximation Theory.

1. Suppose (E, T) is a normed linear space and $p(x) = \|x\|$. Then Theorem 1 contains a result of Hadzic ([4], Theorem 4) and Theorem 2 gives a result due to Prolla [8]. In case $g(x) = x$, then we get from Theorem 1 a well-known theorem of Ky Fan ([3] Theorem 7), and Theorem 2 yields another famous theorem due to Ky Fan ([2], Theorem 2).

2. Suppose that E is a normed linear space and T is the weak topology of E . If we assume that $p(x) = \|x\|$, and $g(x) = x$ we obtain from Theorem 1 a result due to Sehgal, Singh and Smithson

([10], Theorem 3), and from Theorem 2 a result due to Kapoor ([7], Theorem 2). (Remember that $x \rightarrow \| x \|$ is lower semicontinuous on (E, T)).

3. Analogous results hold if E is a dual normed space and T is the weak $*$ topology of E . (Indeed, $x \rightarrow \| x \|$ is lower semicontinuous also in the weak $*$ topology of E).

4. Suppose that (E, T) is a locally convex topological vector space and P is the family of seminorms which generates T . Since any seminorm $p \in P$ is continuous on (E, T) and $f : (X, T) \rightarrow (E, T)$ continuous obviously imply that $f : (X, T) \rightarrow (E, p)$ is continuous, Theorem 2 contains Theorem 1 of Ky Fan [2] and a result due to Sehgal and Singh [9].

5. Suppose that T is the weak topology of a locally convex topological vector space (E, τ) and P is the family of seminorms which generates τ . Then since every $p \in P$ is lower semicontinuous on (E, T) and $f : (X, T) \rightarrow (E, \tau)$ continuous imply that $f : (X, T) \rightarrow (E, p)$ is continuous, so Theorem 1 contains Theorem 1 of Sehgal, Singh and Smithson [10].

6. Finally, Theorem 1 obviously contains Theorem 4 of Hadzic [4, p. 34]. Theorems 1 and 2 also allow us to obtain coincidence results.

Let us recall the inwardness condition. This condition is weaker than the condition $x \in \partial X \Rightarrow fx \in X$, (∂X stands for boundary of X) and is widely used in order to give fixed point theorems for mappings $f : X \rightarrow E$ ($X \subseteq E$).

For $x \in X$ the inward and outward sets of X at x are defined by

$$I_X(x) = \{x + r(y-x) \in E : y \in X, r > 0\}.$$

$$O_X(x) = \{x - r(p-x) \in E : y \in X, r > 0\}.$$

If x is an internal point then $I_X(x) = O_X(x) = E$.

If $f: X \rightarrow E$ and $fx \in I_X(x) (O_X(x))$ for each $x \in X$ then f is said to satisfy inward (outward) condition.

In case X is a convex set this condition may be given as follows (see Massa [7]).

A map $f: X \rightarrow E$ is an inward map if

$$(5) \quad x \in X \Rightarrow \{y \in E : y = rx + (1-r)fx, 0 \leq r < 1\} \cap X \neq \phi.$$

In the case X is a convex set and g is onto X , we introduce the following more general definition:

A map $f: X \rightarrow E$ is said to be g -inward if

$$(6) \quad x \in X \Rightarrow \{y \in E : y = rgx + (1-r)fx, 0 < r < 1\} \cap X \neq \phi.$$

The following theorems hold:

Theorem 3 (Theorem 4).

Suppose that the hypotheses of Theorem 1 (Theorem 2) are satisfied and g is onto X . If f is a g -inward map, then there exists a point $u \in K$ such that:

$$(7) \quad p(gu-fu) = 0.$$

If moreover, $x \in (g \cdot f)(K)$ and $p(x)=0$ imply that $x=0$, then we have $fu = gu$.

Proof:

Let us consider a point u which satisfies (4). If $fu \in X$, then u satisfies

(7). If $fu \notin X$, let $r, 0 < r < 1$ such that $rgu + (1-r)fu \in X$. Then (4) gives

$$\begin{aligned} 0 &\leq p(gu - fu) \\ &\leq p(rgu + (1-r)fu - fu) \\ &\leq rp(gu - fu) \end{aligned}$$

which is absurd if $p(gu - fu) \neq 0$. Hence (7) holds.

Remark :

If E is a complex vector space and X is a convex set the inwardness condition (5) may be weakened as follows: f is an inward map if (5¹) $x \in X$ implies that $\{y \in E : y = rx + (1-r)fx, |r| < 1\} \cap X \neq \phi$. (In case E is a real vector space, condition (5¹) is easily shown to be equivalent to condition (5)).

The g -inwardness condition (6) may be weakened in the same way. The map f is g -inward if

$$(6^1) \ x \in X \rightarrow \{y \in E : y = rgx + (1-r)fx, |r| < 1\} \cap X \neq \phi.$$

If f satisfies (6¹), Theorem 3 (Theorem 4) still holds.

In order to prove that our results contain the more general theorems of coincidence and fixed points known till now in normed spaces, locally convex spaces are topological vector spaces, we will use the following

Lemma :

Let E be a topological vector space with sufficiently many linear functionals. If $H \subset E$ is a weakly compact set and $0 \notin H$, then there exists a seminorms p , continuous in the weak topology of E , such that $p(x) \neq 0$ for all $x \in H$.

Proof :

For every $x \in H$ there exists a continuous linear functional $x'_x \in E'$ and a w -open neighborhood $N(x)$ of x such that $|x'_x(y)| > 0$ for all $y \in N(x)$.

Let $\{N(x_i)\}$, $i = 1, 2, \dots, n$ be a finite subcovering of the covering $\{N(x)\}_{x \in H}$ of H . Then the seminorm $p(x) = \sum_{i=1}^n |x'_{x_i}(x)|$ is positive for every $x \in H$. Moreover, for every $x \in E$ and $\epsilon > 0$ if $y \in N$, where $N = \{z \in E : |x'_{x_i}(z-x)| < \frac{\epsilon}{n}, i=1, 2, \dots, n\}$, we have $p(y) < \epsilon$. Then p is continuous in the weak topology of E .

Since a strongly compact subset of E is also weakly compact and since the seminorm p of the Lemma is obviously continuous in the strong topology of E , it is easy to see that Theorem 3 contains as a particular case. Theorem 5 of Hadzic [4] and Theorem 4 contains the Theorem of Kaczynski [5]. (Recall remark after Theorem 3 and assume $g(x) = x$).

We may also obtain new results. As an example, Theorem 4 and Lemma give the following :

Theorem (5) :

Let (E, τ) be a topological vector space with sufficiently many linear functionals, $K \subset E$, a non-empty, convex, τ -compact set, T the weak topology of E and $f: (X, \tau) \rightarrow (E, T)$ be a continuous inward map. Then f has a fixed point in K .

Theorem 6.

Let (E, τ) be a topological vector space with sufficiently many linear functionals, T the weak topology of E , $K \subset E$ a non-empty,

convex, T -compact set, $f: (K, T) \rightarrow (E, \tau)$ a continuous inward map. Then f has a fixed point in K .

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A FIXED POINT THEOREM FOR CONTRACTION TYPE MAPPINGS IN CONVEX SPACES

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ABSTRACT

The result herein provides a common fixed point theorem for a pair of contraction type mappings with respect to a bounded closed and convex subset of a locally convex space. Theorem 1 below extends an earlier result of Dubinsky and the authors ,

Let E be a separated locally convex topological vector space and M a nonempty convex subset of E containing the origin 0 . Let B be a closed, bounded and convex subset of S and $\phi : [0, \infty) \rightarrow [0, \infty)$ a mapping with $\phi(t) \leq t$ for t close to zero . Let $C(E)$ denote the nonempty closed subsets of E .

Definition .

Let $x_0 \in E$ and for $i \in \{1, 2\}$, let $f_i : x_0 + M \rightarrow C(E)$ be a pair of mappings . The pair (f_1, f_2) is a ϕ -contraction iff for each x, y in $x_0 + M$ with $y - x \in \lambda B$ for some $\lambda \geq 0$ and $u \in f_i(x)$ implies that $(f_j(y) - u) \cap \phi(\lambda) B \neq \emptyset, i \neq j$. Note that the definition implies

that for $u \in f_j(x)$ there exists a $v \in f_j(y)$, $j \neq i$ with $v-u \in \phi(\lambda) B$.

Theorem 1.

Let E be sequentially complete and for $i=1, 2$, let $f_i: x_0 + M \rightarrow C(E)$ be a pair mappings. If the pair (f_1, f_2) is a ϕ -contraction and for some k , $0 \leq k < 1$

$$(1) \quad (f_1(x_0) - x_0) \cap kB \neq \phi$$

$$(2) \quad \sum_{p=0}^{\infty} \phi^p(k) = a \leq 1 \text{ where } \phi^0(k) = k,$$

then f_1, f_2 have a common fixed point, that is there exists a $u \in x_0 + M$ with $u \in f_1(u) \cap f_2(u)$.

Proof .

First, we show by induction the existence of a sequence $\{x_n\}$ satisfying

$$(a) \quad x_n \in x_0 + M \text{ for each } n,$$

$$(b) \quad x_n \in \begin{cases} f_1(x_{n-1}) & \text{if } n \text{ is an odd integer} \\ f_n(x_{n-1}) & \text{if } n \text{ is even} \end{cases}$$

$$(c) \quad x_n - x_{n-1} \in \phi^{n-1}(k) B \text{ for each } n.$$

Clearly by (1) there exists a $x_1 \in f_1(x_0)$ with $x_1 - x_0 \in kB \subseteq M$. Assume that for some fixed n , x_i , $i=1, 2, \dots, n$ have been obtained satisfying (a), (b) and (c) above. Without loss of generality we may assume that n is an even integer. This implies that

$$x_n \in (x_0 + M) \cap f_2(x_{n-1}) \text{ and}$$

$$x_n - x_{n-1} \in \phi^{n-1}(k) B \subseteq M.$$

since $x_n \in f_2(x_{n-1})$, it follows by definition that there exists a $x_{n+1} \in f_1(x_n)$ with $x_{n+1} - x_n \in \phi^n(k) B$. To complete the induction, we need to show that $x_{n+1} \in x_0 + M$. To prove this, note that since B is convex

$$(3) \quad x_{n+1} - x_0 = \sum_{p=0}^n (x_{p+1} - x_p) \in \sum_{p=0}^n \phi^p(k) B$$

$$\subseteq \left(\sum_{p=0}^n \phi^p(k) \right) B \subseteq aB \subseteq M.$$

Thus, $x_{n+1} \in x_0 + M$. This completes the proof of the existence of the stated sequence $\{x_n\}$.

Now, we show that the sequence $\{x_n\}$ thus constructed is in fact a Cauchy sequence. Let V be an arbitrary convex neighborhood of the origin. Since, B is a bounded set, choose an $\varepsilon_0 > 0$ such that $\varepsilon B \subseteq V$ and also satisfies $\phi(\varepsilon) \leq \varepsilon$ for $0 \leq \varepsilon \leq \varepsilon_0$. Let N be any

integer for which $\sum_{p=N}^{\infty} \phi^p(k) \leq \varepsilon_0$.

Let $\lambda_n = \sum_{p=n}^{\infty} \phi^p(k)$. For $m \geq n \geq N$,

$$(4) \quad x_m - x_n = \sum_{p=n}^{m-1} (x_{p+1} - x_p) \in \left(\sum_{p=n}^{m-1} \phi^p(k) \right) B \subseteq \lambda_n B \subseteq V.$$

This proves that the sequence $\{x_n\}$ is Cauchy in E . Consequently, there is a $u \in E$ such that $x_n \rightarrow u$. Further, since B is closed, it follows from (3) that $u - x_0 \in aB$, that is $u \in x_0 + M$ and (4) implies that for all $n \geq N$, $u - x_n \in \lambda_n B$. In particular for $n \geq N$

a) $u - x_{2n} \in \lambda_{2n} B$ and b) $u - x_{2n+1} \in \lambda_{2n+1} B$.

We show that (a) implies that $u \in f_2(u)$. Since $x_{2n+1} \in f_1(x_{2n})$ it follows by (a) that for each $n \geq N$, there is a $u_n \in f_2(u)$ satisfying

$$u_n - x_{2n+1} \in \phi(\lambda_{2n})B \subseteq \lambda_{2n}B.$$

Since $\lambda_{2n} \rightarrow 0$ as $n \rightarrow \infty$, it follows as in (4) that $u_n - x_{2n+1} \rightarrow 0$, that is $u_n \rightarrow u$. However, $u_n \in f_2(u)$ and $f_2(u)$ is closed. Consequently, $u \in f_2(u)$.

In a similar way, it follows from (b) that $u \in f_1(u)$.

Thus $u \in f_1(u) \cap f_2(u)$.

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UNIFORM SUMMABILITY- Δ OF ORTHONORMAL EXPANSIONS OF SUMMABLE FUNCTIONS

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ABSTRACT

In this paper, we establish necessary and sufficient conditions for the uniform summability- Δ , of the orthonormal expansions of summable functions.

Let $f(x)$ be an integrable function and $\{\phi_v(x)\}$, $v=0, 1, 2, \dots$ be an orthonormal system in $[0, 1]$.

Let

$$f(x) \sim \sum_{v=0}^{\infty} c_v \phi_v(x),$$

where

$$c_v = \int_0^1 f(t) \phi_v(t) dt \quad (v = 0, 1, 2, \dots),$$

be an orthonormal expansion of $f(x)$ in the system $\{\phi_v(x)\}$.

Let $\{\lambda_v\}$ be an infinite sequence of real numbers,

If the series

$$\sum_{v=0}^{\infty} \lambda_v c_v \phi_v(x)$$

converges uniformly in x , then $\{\lambda_v\}$ is called a sequence of uniform convergence factors of orthonormal expansions of $f(x)$.

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If

$$f(x) \sim \sum_{v=0}^{\infty} c_v \phi_v(x)$$

then

$$\begin{aligned} S_n(f; x) &= \sum_{v=0}^n \lambda_v c_v \phi_v(x) = \int_0^1 f(t) \left(\sum_{v=0}^n \lambda_v \phi_v(x) \phi_v(t) \right) dt \\ &= \int_0^1 f(t) K_n(x, t) dt \end{aligned}$$

where

$$K_n(x, t) = \sum_{v=0}^n \lambda_v \phi_v(x) \phi_v(t),$$

In this paper we consider an orthonormal system $\{\phi_v(x)\}$ closed in $C[0, 1]$, the class of continuous functions on $[0, 1]$, and $|\phi_v(x)| < A_v$, independent of x .

In [3] we proved the following theorem ,

Theorem 1. A necessary and sufficient condition that $\{\lambda_v\}$ be a sequence of uniform convergence factors of orthonormal expansions of $f(x) \in L$ (summable functions), is that there exists an M such that

$$\text{ess sup}_{0 \leq t \leq 1} |K_n(x, t)| < M$$

for all x and n .

Here ess sup lends a symmetry to the result and actually it is not needed, since $|\phi_v(x)| < A_v$ for all v .

Now let

$$\|\lambda_{v\mu}\| \quad (v, \mu = 0, 1, 2, \dots)$$

be an infinite matrix of real elements .

If

$$\Lambda_{\mu, x} (f) = \sum_{v=0}^{\infty} \lambda_{v\mu} c_v \phi_v(x)$$

is uniformly convergent in x , for each $\mu = 0, 1, 2, \dots$ and

$$\lim_{\mu \rightarrow \infty} \Lambda_{\mu, x} (f) = \Lambda_x (f)$$

exists uniformly in x , then orthonormal expansions of $f(x)$ are said to be uniformly summable- Λ .

The Λ -summation method is not necessarily regular, *i. e.*, every convergent series is not summed by it to its sum in the sense of convergence.

We note if

$$f(x) \sim \sum_{v=0}^{\infty} c_v \phi_v(x)$$

then

$$\begin{aligned} \Lambda_{n, \mu, x} (f) &= \sum_{v=0}^n \lambda_{v\mu} c_v \phi_v(x) = \int_0^1 f(t) \left(\sum_{v=0}^n \lambda_{v\mu} \phi_v(x) \phi_v(t) \right) dt \\ &= \int_0^1 f(t) K_{n, \mu} (x, t) dt \end{aligned}$$

where

$$K_{n, \mu} (x, t) = \sum_{v=0}^n \lambda_{v\mu} \phi_v(x) \phi_v(t).$$

We may write

$$\begin{aligned}\Lambda_{\mu, x}(t) &= \sum_{v=0}^{\infty} \lambda_{v\mu} c_v \phi_v(x) = \lim_{n \rightarrow \infty} \Lambda_{n, \mu, x}(f) \\ &= \lim_{n \rightarrow \infty} \int_0^1 f(t) K_{n, \mu}(x, t) dt.\end{aligned}$$

Formally, we write

$$\Lambda_{\mu, x}(f) = \int_0^1 f(t) \left(\sum_{v=0}^{\infty} \lambda_{v\mu} \phi_v(x) \phi_v(t) \right) dt = \int_0^1 f(t) K_{\mu}(x, t) dt.$$

We do not mean

$$K_{\mu}(x, t) = \sum_{v=0}^{\infty} \lambda_{v\mu} \phi_v(x) \phi_v(t)$$

in the sense of ordinary convergence, but rather a formal relationship which can be made precise by a suitable definition for the sum of the series.

We first need a lemma.

Lemma 1. If $\{g_n(x)\}$ is a sequence in B , the space of measurable bounded functions, such that for every $f(x) \in L$

$$\lim_{n \rightarrow \infty} \int_0^1 f(x) g_n(x) dx$$

exists, and $|g_n(x)| \leq C$ almost everywhere, then there exists a $g(x) \in B$, $|g(x)| \leq C$ almost everywhere such that

$$\lim_{n \rightarrow \infty} \int_0^1 f(x) g_n(x) dx = \int_0^1 f(x) g(x) dx$$

for every $f(x) \in L$.

Proof : If $\{g_n(x)\} \in B$, then $\{g_n(x)\} \in L$ and by hypothesis

$$\lim_{n \rightarrow \infty} \int_0^1 f(x) g_n(x) dx$$

exists for every $f(x) \in B$:

Therefore, $\{g_n(x)\}$ converges weakly in L and since L is weakly complete ([2], p. 240) there exists a $g(x) \in L$ such that $g_n(x)$ converges weakly to $g(x)$ in L .

Now suppose

$$|g(x)| > C$$

on a set H of positive measure, i. e., $m(H) > 0$.

Let $X(x)$ be the characteristic function of H ,

Then

$$\lim_{n \rightarrow \infty} \int_0^1 X(x) (g_n(x) - g(x)) dx = \lim_{n \rightarrow \infty} \int_H (g_n(x) - g(x)) dx > 0,$$

which contradicts the weak convergence of $g_n(x)$ to $g(x)$ in L ,
Hence

$$|g(x)| \leq C.$$

Now let $f(x) \in L$, we must show that

$$\lim_{n \rightarrow \infty} \int_0^1 f(x) g_n(x) dx = \int_0^1 f(x) g(x) dx.$$

For every $\varepsilon > 0$, we decompose $f(x)$ such that

$$f(x) = f_1(x) + f_2(x)$$

where

$$|f_1(x)| \leq M$$

and

$$\int_0^1 |f_2(x)| dx < \varepsilon.$$

then

$$\begin{aligned} \left| \int_0^1 f(x) (g_n(x) - g(x)) dx \right| &= \left| \int_0^1 (f_1(x) + f_2(x)) (g_n(x) - g(x)) dx \right| \\ &\leq \left| \int_0^1 f_1(x) (g_n(x) - g(x)) dx \right| + \left| \int_0^1 f_2(x) (g_n(x) - g(x)) dx \right| \\ &< \varepsilon + 2C\varepsilon \end{aligned}$$

for sufficiently large n .

Hence

$$\lim_{n \rightarrow \infty} \int_0^1 f(x) g_n(x) dx = \int_0^1 f(x) g(x) dx$$

for every $f(x) \in L$.

The proof of Lemma 1 is complete.

Now, from Lemma 1, there exists a $K_\mu(x, t) \in B$ such that

$$\lim_{n \rightarrow \infty} \int_0^1 f(t) K_{n,\mu}(x, t) dt = \int_0^1 f(t) K_\mu(x, t) dt$$

for every $f(t) \in L$.

We note that $\{K_{n,\mu}(x, t)\}$ meets the requirements of Lemma 1 by

Theorem 1.

Now we get our main theorem .

Theorem 2. The necessary and sufficient conditions for the uniform summability- Λ of orthonormal expansions of $f(x) \in L$, are

$$(1) \quad \lim_{\mu \rightarrow \infty} \lambda_{\nu\mu}$$

exists for each $\nu = 0, 1, 2, \dots$

$$(2) \quad \begin{array}{c} \text{ess sup} \\ 0 \leq t \leq 1 \end{array} |K_{n,\mu}(x, t)| < M_\mu \quad (\mu = 0, 1, 2, \dots)$$

independent of n and x .

$$(3) \quad \begin{array}{c} \text{ess sup} \\ 0 \leq t \leq 1 \end{array} |K_\mu(x, t)| < M$$

independent of μ and x .

To prove the theorem we use a lemma of Banach-Steinhaus type ([3]) .

Lemma 2. Let $\{A_{n,x}(f)\}$ be a sequence of linear functionals defined on the Banach space X , and Y be an everywhere dense subset of X .

In order that $\lim_{n \rightarrow \infty} A_{n,x}(f)$ exists uniformly in x (x belonging to a set E) for every $f \in X$, it is a necessary and sufficient that

(A) there exists an N and an M such that

$$\|A_{n,x} - A_{m,x}\| < M$$

for all x , and $n, m > N$.

(B) $\lim_{n \rightarrow \infty} A_{n,x}(g)$ exists uniformly in x , for every $g \in Y$.

Proof of Theorem 2.

Clearly $\Lambda_{\mu,x}(f) = \lim_{n \rightarrow \infty} \Lambda_{n,\mu,x}(f)$ is a continuous linear functional on L .

Now for each $\mu = 0, 1, 2, \dots$, the sequence $\{\lambda_{v,\mu}\}$ be a sequence of uniform convergence factors of orthonormal expansions of $f(x) \in L$ the condition (2) is necessary and sufficient by Theorem 1.

We show that condition (B) of Lemma 2 is satisfied in the theorem. Since $\{\phi_v(x)\}$ is closed in C , the finite linear combinations

$$g(x) = \sum_{p=0}^k \gamma_p \phi_p(x),$$

are dense in C and hence dense in L .

From the orthonormality of $\{\phi_v(x)\}$ we get

$$\begin{aligned} \Lambda_{\mu,x}(g) &= \lim_{n \rightarrow \infty} \Lambda_{n,\mu,x}(g) \\ &= \lim_{n \rightarrow \infty} \int_0^1 \left(\sum_{p=0}^k \lambda_p \phi_p(t) \right) \left(\sum_{v=0}^n \lambda_{v,\mu} \phi_v(x) \phi_v(t) \right) dt \\ &= \sum_{p=0}^k \gamma_p \lambda_{p,\mu} \phi_p(x). \end{aligned}$$

Hence if

$$\lim_{\mu \rightarrow \infty} \Lambda_{\mu,x}(g)$$

exists uniformly in x , then

$$\lim_{\mu \rightarrow \infty} \lambda_{v,\mu}$$

exists for each $v=0, 1, 2, \dots$

So condition (B) of Lemma 2 implies condition (1) of the theorem.

Now consider

$$\begin{aligned} \left| \Lambda_{\mu', x}(g) - \Lambda_{\mu'', x}(g) \right| &= \left| \lim_{n \rightarrow \infty} \int_0^1 g(t) (K)_{n, \mu'}(x, t) - \right. \\ &\quad \left. K_{n, \mu''}(x, t) dt \right| = \left| \sum_{p=0}^k \gamma_p (\lambda_{p\mu'} - \lambda_{p\mu'') \phi_p(x) \right| \\ &\leq \sum_{p=0}^k \left| \gamma_p \right| \left| \lambda_{p\mu'} - \lambda_{p\mu''} \right| \left| \phi_p(x) \right| \\ &\leq \sum_{p=0}^k \left| \gamma_p \right| \left| \lambda_{p\mu'} - \lambda_{p\mu''} \right| A_p. \end{aligned}$$

Hence, if

$$\lim_{\mu \rightarrow \infty} \lambda_{v\mu}$$

exists for each $v = 0, 1, 2, \dots$ then

$$\lim_{\mu \rightarrow \infty} \Lambda_{\mu, x}(g)$$

exists uniformly in x , for every g belonging to dense subset of L .

Thus condition (B) of Lemma 2 is equivalent to condition (1) of the theorem.

We note

$$\left\| \Lambda_{\mu, x} \right\| = \text{ess sup}_{0 \leq t \leq 1} \left| K_{\mu}(x, t) \right|.$$

We show that condition (A) of Lemma 2 is equivalent to uniform boundedness of norm, i. e.,

$$\| \Lambda_{\mu, x} \| \leq M$$

for all μ and x , hence equivalent to condition (3) of the theorem. Assume

$$\| \Lambda_{\mu, x} \| < M$$

for all x , and $\mu > N$.

$$\text{We get } \| \Lambda_{\mu'} - \Lambda_{\mu''}, x \| \leq \| \Lambda_{\mu'}, x \| + \| \Lambda_{\mu''}, x \| < 2M$$

for all x , and $\mu', \mu'' > N$.

So condition (1) of the theorem implies condition (A) of Lemma 2. Now since $|\phi_v(x)| < A_v$, independent of x ,

$$|K_{\mu}(x, t)| < C_{\mu}$$

independent of x and t .

Consider

$$\begin{aligned} \left| \Lambda_{\mu, x}(f) \right| &= \left| \int_0^1 f(t) K_{\mu}(x, t) dt \right| \leq \|f\|_{\text{ess}} \sup |K_{\mu}(x, t)| \\ &\leq \|f\| C_{\mu}. \end{aligned}$$

Therefore,

$$\| \Lambda_{\mu, x} \| \leq C_{\mu}.$$

Now condition (A) of Lemma 2 gives

$$\| \Lambda_{\mu'}, x - \Lambda_{\mu''}, x \| < M$$

for all x , and $\mu', \mu'' > N$.

$$\begin{aligned} \text{So } \|\Lambda_{\mu', x}\| &\leq \|\Lambda_{\mu'', x}\| + M \\ &\leq C_{\mu''} + M = M' \end{aligned}$$

for all x , and $\mu' > N'$.

Hence, condition (A) of Lemma 2 is equivalent to condition (3) of our theorem.

Thus we have shown that conditions (1) and (3) of our theorem are necessary and sufficient for the uniform convergence of $\Lambda_{\mu, x}(f)$,

The proof of Theorem 2 is complete.

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COLLINEATION INDUCED INCIDENCE RELATIONS*

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1. INTRODUCTION

The triangle has associated with it a number of wellknown remarkable points. While each of these possess individual properties important in the geometry of the triangle, certain subsets share the further property of lying on the same line. Some of these, such as the Euler line which contains the centroid, circumcentre, orthocentre, and nine-point centre of the given triangle, have been known for some time while others, described later, are more recent. In this paper, we apply simple collineations to some known incidence relations and obtain others believed to be new. Interestingly, all of the latter except one, involve a seemingly little known point, the "middlespoint" that appears in a paper by C. H. von Nagel published in 1836.

As a secondary objective, we attempt to demonstrate how classical Euclidean geometry of the plane may be enriched and made more interesting by the use of other techniques, in this case, those of projective geometry. In return, material that is essentially Euclidean, but which has a natural link with real projective geometry, can help make more concrete, and hence more appealing, the abstract area of general projective geometry. Since the latter, with its attention to

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rigor, generally takes precedence over the Euclidean in present-day curricula, we shall emphasize the second interaction.

2. Background.

In the Euclidean plane E^2 , any two points determine a unique line. This also holds in the projective plane P^2 where we have, additionally, the dual statement every pair of lines determine a unique point and parallelism is not an issue. While this more general setting can not be handled by the usual Cartesian coordinate system, a related system works quite nicely.

Consider the equations

$$x = \frac{x_1}{x_3}, \quad y = \frac{x_2}{x_3}$$

where $x_i, i=1, 2, 3$, are real and $x_3 \neq 0$, then each ordered triple (x_1, x_2, x_3) has the ordered pair (x, y) as a unique mate in E^2 when only the ratios $x_1 : x_2 : x_3 = x : y : 1$ are considered. The triple (x_1, x_2, x_3) thus represents the homogeneous coordinates of a point p in the projective plane. Since all non-zero scalar multiples of a given triple represent the same point, we have the added convenience of working with integer coordinates. For example, the mate of the point $\left(\frac{1}{2}, -\frac{2}{7}\right)$ in E^2 has coordinates $\left(\frac{1}{2}, -\frac{2}{7}, 1\right) = (7, -4, 14)$ in P^2 . The points $(x_1, x_2, 0)$ have no mates in E^2 and thus constitute the line at infinity, $l_\infty : x_3 = 0$, in P^2 .

There is, in general, no projective counterpart for the Cartesian distance formula thus projective geometry with such non-metric notions as straightness and tangency. One can include parallelism in this non-metric framework by singling out the line at infinity and

defining a pencil of parallel lines as one with its centre on this line. This geometry, which is sort of in between Euclidean and projective, is called affine geometry.

A fundamental property of the projective plane is that it has a collineation structure, *i.e.* there exists a map called a collineation, that preserves straight lines (point ranges) and line pencils. This map, defined by the rule

$$\rho x'_i = \sum_j a_{ij} x_j; i, j = 1, 2, 3, \quad (1)$$

where the determinant $|a_{ij}| \neq 0$ and ρ is a non-zero constant, maps collinear points to image sets which are also collinear. When dealing with the geometry of the triangle, one needs to use the measures of the sides and angles of the given triangle. Two systems of specialized homogeneous coordinates that do this, while at the same time retain all the advantages of the general homogeneous type, are the trilinear (or normal) system and the barycentric (or areal) system.

We now provide a brief description of each of these systems, see [5] or [7]. In the trilinear case, the coordinates of a point $P(x_1, x_2, x_3)$ in the plane are proportional to the signed distances d_1, d_2, d_3 of P from the sides of a given triangle of reference $A_1A_2A_3$. The vertices of this triangle have coordinate representations $A_1(1, 0, 0)$, $A_2(0, 1, 0)$, $A_3(0, 0, 1)$ and the unit point of the system is the incentre $I=(r, r, r) = (1, 1, 1)$, where r is the inradius of $A_1A_2A_3$. The distance $d_i, i=1, 2, 3$ is positive if p is in the half-plane determined by I and $A_3A_k, i \in \{j, k\}$, and negative otherwise. The excentre I_1 opposite vertex A_1 , for example, has coordinates $(-1, 1, 1)$ or its projective equivalent $(1, -1, -1)$. In the barycentric case, the coordinates of $P(y_1, y_2, y_3)$ are proportional to the signed areas of the triangle $PA_2A_3, PA_3A_1, PA_1A_2$ with the signs being determined as in the trilinear case with

the unit point of the system being the centroid S . The line at infinity is the unit line $x_1 + x_2 + x_3 = 0$, while for the trilinear case it is $a_1x_1 + a_2x_2 + a_3x_3 = 0$, where $a_i, i = 1, 2, 3$ are the measures of the sides of the reference triangle. We shall use this system for the remainder of the paper.

Incidence relations are readily determined by such a system. Consider the points $P_1 (x_1, x_2, x_3)$, $P_2 (y_1, y_2, y_3)$, $P_3 (z_1, z_2, z_3)$ respectively; then P_1, P_2, P_3 are collinear if and only if the determinantal equation

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix} = 0 \quad (2)$$

is true.

For concurrency of lines, there exists a dual procedure. If, for example, we represent the line $l_1 : u_1x_1 + u_2x_2 + u_3x_3 = 0$ by its "line" or "pluckerian" coordinates $l_1 (u_1, u_2, u_3)$. see [6]. then $l_1, l_2 (v_1, v_2, v_3)$, and $l_3 (w_1, w_2, w_3)$ are concurrent if and only if

$$\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = 0. \quad (3)$$

In Table I, we list several of the special points in the plane associated with the triangle of reference $A_1A_2A_3$ along with their coordinate representations.

Table 1

Point	Symbol	Trilinear Coordinatea
Incentre	I	$(1, 1, 1)$
Excentre	I_{A_1}	$(-1, 1, 1)$
Centroid	S	$(\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3})$
Circumcentre	O	$(\cos \alpha_1, \cos \alpha_2, \cos \alpha_3)$
Orthocentre	H	$(\frac{1}{\cos \alpha_1}, \frac{1}{\cos \alpha_2}, \frac{1}{\cos \alpha_3})$
Lemoine point	K	(a_1, a_2, a_3)
Nagel point	G	$(\frac{s-a_1}{a_1}, \frac{s-a_2}{a_2}, \frac{s-a_3}{a_3}), s = \frac{a+b+c}{2}$
Gergonne point	N	$(\frac{1}{a_1(s-a_1)}, \frac{1}{a_2(s-a_2)}, \frac{1}{a_3(s-a_3)})$
Middlespoint	M	$(s-a_1, s-a_2, s-a_3)$

The derivation of these coordinates is especially easy when the segments, which the concurrent lines through the given point and the vertices of $A_1A_2A_3$ cut the opposite sides, are known. Consider the point P and the pencil of lines $A_1PD_1, A_2PD_2, A_3PD_3$, where D_1 is on A_2A_3 , etc., see figure 1, then for the point

$d_1, d_2 : d_3 = \frac{D_1A_3 \sin \alpha_3}{A_2D_1 \sin \alpha_2}$, where $\alpha_1, \alpha_2, \alpha_3$ denote the measures of the angles at A_1, A_2, A_3 respectively, see Table 1. Using the well known expressions $a_2 = 2R \sin \alpha_2$ and $a_3 = 2R \sin \alpha_3$, where R is the circumradius of $A_1A_2A_3$, which relate the measures of the sides

and angles of the given triangle, we obtain

$$D_1 = (0, a_3 \overline{D_1 A_3}, a_2 \overline{A_2 D_1}) \quad (4)$$

as the trilinear coordinates of D_1 . Since the coordinates for D_2 and D_3 may be derived in a similar manner, the coordinates of P are immediate.

To illustrate the idea, we consider three points from table 1 that might not be widely known and indicate briefly how to derive their corresponding coordinates. All three are readily derived as above since the positions of the $D_i, i=1, 2, 3$, in each case are known. We first consider the Lemoine or symmedian point K where the D_i 's are such that

$$\frac{A_j D_i}{D_i A_k} = \frac{a_k^2}{a_j^2}, \quad i, j, k = 1, 2, 3.$$

In this case, $D_1 = (0, a_2, a_3)$ with similar representations for D_2 and D_3 , hence $K = (a_1, a_2, a_3)$.

Also fitting this picture are the Gergonne and Nagel points G and N shown in figures 2 and 3 respectively.

The coordinates of each of these points now follow readily since for G , $A_2 \overline{D_1} = s - a_2$ and for N , $\overline{A_2 D_1} = s - a_3$, etc.

We now come to the last point in table 1 and one which plays a key role in this paper. see figure 4.

Let S_1, S_2, S_3 denote the midpoints of $\overline{A_2 A_3}, \overline{A_3 A_1}, \overline{A_1 A_2}$ respectively, then the lines $I_1 S_1, I_2 S_2, I_3 S_3$ are concurrent at the midespoint (mittenpunkt), N of the given triangle, see Nagel [4].

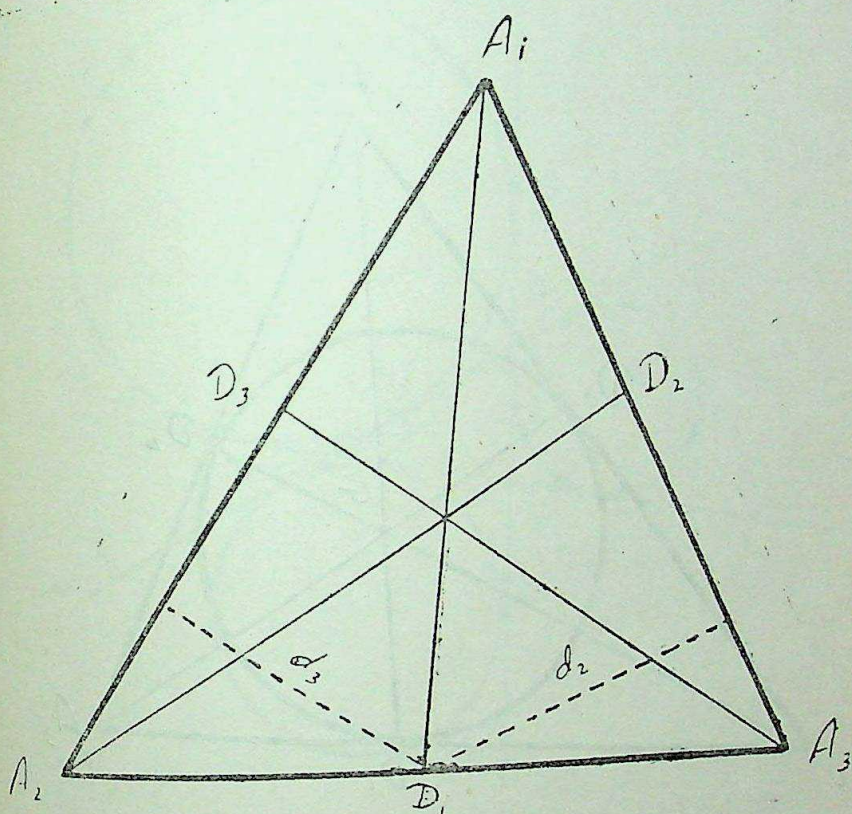


FIGURE 1

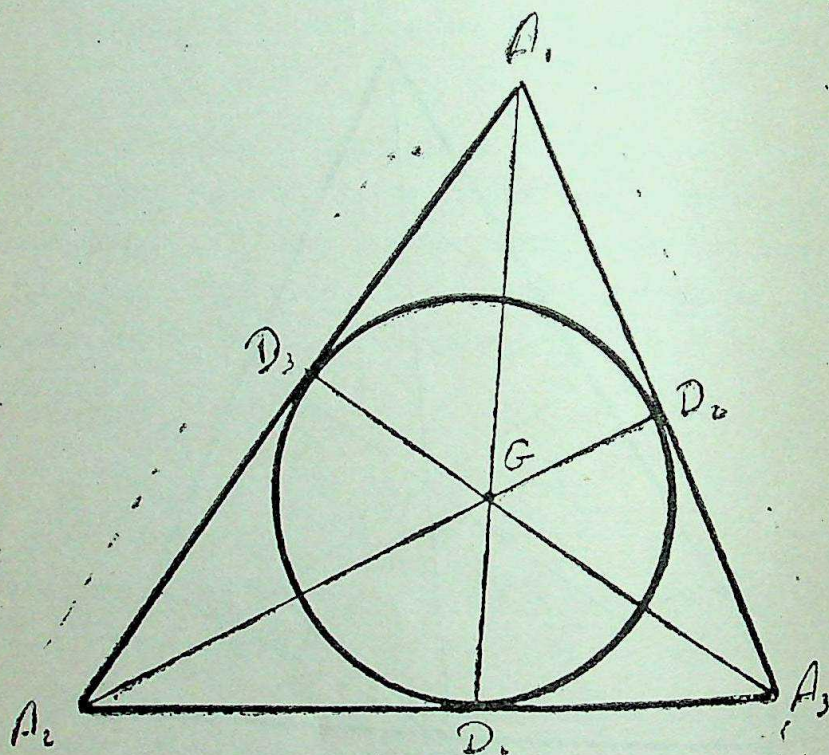


FIGURE 2

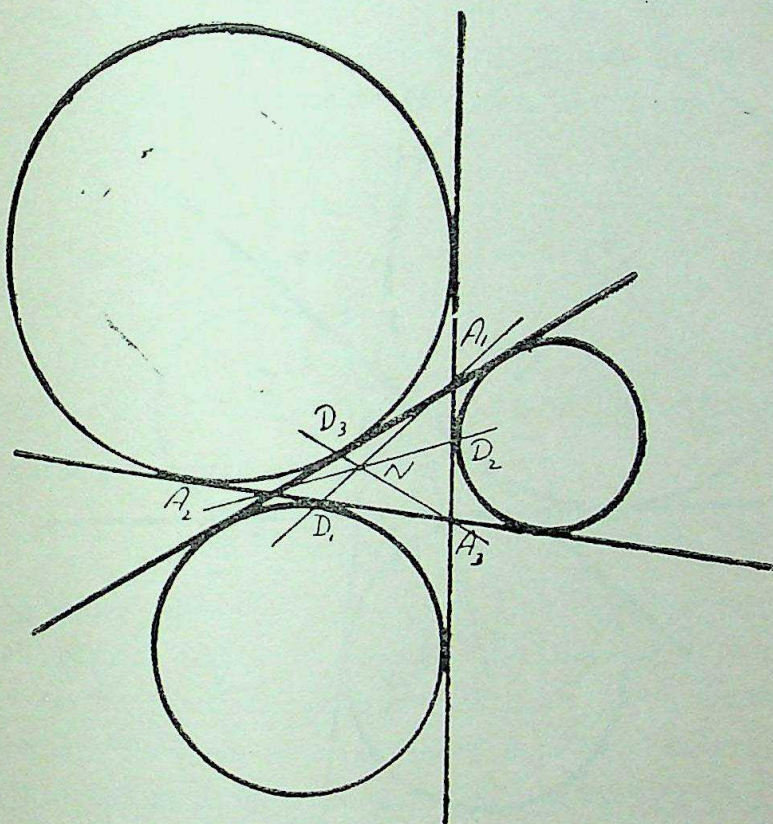


FIGURE 3

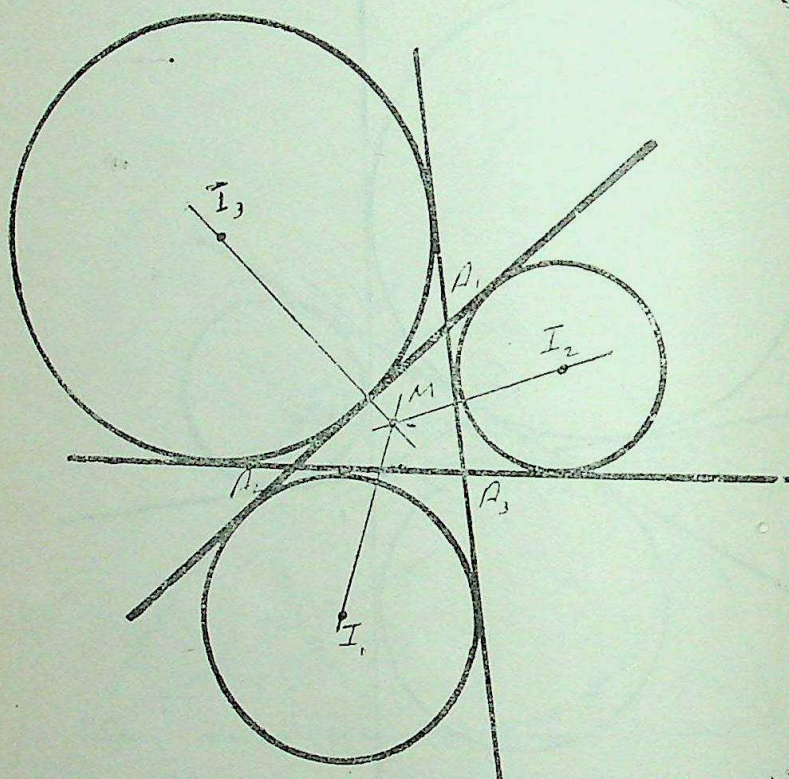


FIGURE 4

Again, the derivation of the coordinates is straightforward. The lines I_1S_1 , I_2S_2 , I_3S_3 are easily seen to have pluckerian coordinates $(a_2 - a_3, a_2, -a_3)$, $(-a_1, a_3 - a_1, a_3)$, $(a_1, -a_2, a_1 - a_2)$, consequently, the concurrency property follows immediately by (3). For the coordinates of M , consider the determinantal equation

$$\begin{vmatrix} u_1 & u_2 & u_3 \\ a_2 - a_3 & a_2 & -a_3 \\ -a_1 & a_3 - a_1 & a_3 \end{vmatrix} = 0, \quad (5)$$

where u_1, u_2, u_3 are the coordinates of a general line in the plane.

Expanding, we obtain the pencil.

$$(a_2 + a_3 - a_1)u_1 + (a_2 + a_1 - a_2)u_2 + (a_1 + a_2 - a_3)u_3 = 0 \quad (6)$$

with centre $M = (s - a_1, s - a_2, s - a_3)$. This interesting point, which perhaps derives its name from the fact that its construction involves "middles" *i. e.* centres of circles and the midpoints of line segments, seems to have been discovered by Nagel and, surprisingly, does not appear to have found its way into the popular literature dealing with the geometry of the triangle. While some points may be "born to blush unseen" either because of a complicated construction or a cumbersome coordinate representation, this is certainly not the case with the midpoint. The only other paper known to us that deals with this point is a paper by Baptist [1], soon to appear, which contains, in particular, proposition 1 below.

Other special points may be obtained from the known ones by applying two quadratic transformations. As will be seen, these turn out to be involutions that are well-defined for all points in the

plane except those incident with the triangle of reference, The first of these is the "isogonal" transformation defined as follows.

If the lines $A_i D_i'$, $i=1, 2, 3$, through P are each reflected in the angle bisector through the same vertex, the reflected lines $A_i D_i'$ are concurrent at P^g , the isogonal conjugate of P , see [3].

It is an immediate consequence of the definition that if P has trilinear coordinates (x_1, x_2, x_3) , then P^g has coordinates

$$\left(\frac{1}{x_1}, \frac{1}{x_2}, \frac{1}{x_3} \right) = (x_2 x_3, x_3 x_1, x_1 x_2). \text{ Two such sets,}$$

immediately seen from Table 1, are $\{O, H\}$ and $\{S, K\}$.

The second transformation is called "isotomic" and is defined as in the following manner.

If the points D_i' are the reflections of D_i in the midpoints S_i of the sides of the triangles $A_1 A_2 A_3$, the lines $A_i D_i'$ concur at P^t , the isotomic conjugate of P , see [3].

Again, it is straightforward to ascertain that the trilinear coordinates of P^t are $\left(\frac{1}{a_1^2 x_1}, \frac{1}{a_2^2 x_2}, \frac{1}{a_3^2 x_3} \right)$. It is readily seen from table 1 that $\{G, N\}$ is a pair of isotomic conjugates.

3. The results. We are now in a position to derive results believed to be new. As indicated in the title, these, will be obtained by applying certain collineations to known incidence relations stated in the form of propositions. Each of the collineations is defined by a matrix $B = (b_{ij})$; $i, j = 1, 2, 3$, which is diagonal with $b_{11} \cdot b_{22} \cdot b_{33} \neq 0$. In general, this class of transformation forms a group

which fixes the triangle of reference and is sharply transitive on the points of the plane not incident with its sides. Thus, for convenience we represent each collineation by the general diagonal element $b_{ii}, i = 1, 2, 3$ of its defining matrix.

PROPOSITION 1. The incentre, Nagel point, and the centroid of a triangle are collinear, see [3].

If we transform this set by the matrix $b_{ii} = a_i$, we obtain the points (a_1, a_2, a_3) , $(s - a_1, s - a_2, s - a_3)$, $(1, 1, 1)$. Thus, we have derived

RESULT 1. The Lemoine point, the middlespoint and the incentre of a triangle are collinear.

PROPOSITION 2. The isogonal conjugate of the Gergonne point, and the isogonal conjugate of the Nagel point, and the incentre are collinear, see [2].

Here we apply the collineation $b_{ii} = \frac{1}{a_i}$ and obtain the points

$$\left(\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3} \right), (s - a_1, s - a_2, s - a_3), \left(\frac{1}{s - a_1}, \frac{1}{s - a_2}, \frac{1}{s - a_3} \right).$$

We now have

RESULT 2. The middlespoint of a given triangle is collinear with its isogonal conjugate and the centroid.

If we consider again the points of proposition 2 and apply the collineation $b_{ii} = \frac{1}{a_i^2}$ we obtain the points $\left(\frac{1}{a_1^2}, \frac{1}{a_2^2}, \frac{1}{a_3^2} \right)$,

$$\left(\frac{s-a_1}{a_1}, \frac{s-a_2}{a_2}, \frac{s-a_3}{a_3} \right), \left(\frac{1}{a_1(s-a_1)}, \frac{1}{a_2(s-a_2)}, \frac{1}{a_3(s-a_3)} \right),$$

thus we have

RESULT 3. The isotomic conjugate of the incentre, the Nagel point, and the Gergonne point are collinear.

PROPOSITION 3. The Gergonne point, the middlespoint, and the centroid of a triangle are collinear, see [1].

We now apply the same collineation as for result 2 and obtain the points

$$\left(\frac{1}{a_1^2(s-a_1)}, \frac{1}{a_2^2(s-a_2)}, \frac{1}{a_3^2(s-a_3)} \right), \left(\frac{s-a_1}{a_1}, \frac{s-a_2}{a_2}, \frac{s-a_3}{a_3} \right),$$

$$\left(\frac{1}{a_1^2}, \frac{1}{a_2^2}, \frac{1}{a_3^2} \right). \text{ we can now state}$$

RESULT 4. The isotomic conjugate of the middlespoint, the Nagel point, and the isotomic conjugate of the incentre are collinear.

We now summarize these results in Table 2. For brevity, the collineations are again represented by the general diagonal terms of the defining matrices.

Table 2

Known Results	Derived Results
1. $\{I, N, S\}$	a_i $\rightarrow$ $\{K, M, I\}$
2. $\{G^g, N^g, I\}$	$\frac{1}{a}$ $\rightarrow$ $\{M, M^g, S\}$
3. $\{G^g, N^g, I\}$	$\frac{1}{a_i^2}$ $\rightarrow$ $\{I^i, N, G\}$
4. $\{G, M, S\}$	$\frac{1}{a_i}$ $\rightarrow$ $\{M^i, N, I^i\}$

By examining the results in the table, we are able to state two further results. From the incident sets $\{M, M^g, S\}$ and $\{G, M, S\}$, we obtain

RESULT 5. The middlespoint of a triangle and its isogonal conjugate are collinear with the Gergonne point and the centroid.

Similarly, the sets $\{I^i, N, G\}$ and $\{M^i, N, I^i\}$ yield.

RESULT 6. The Nagel and Gergonne points of a triangle are collinear with the isotomic conjugates of the incentre and the middle-point.

4. Conclusion As indicated earlier, material of this sort has a twofold value. Not only does it provide concrete examples that may be used in teaching the more abstract projective geometry but also, it helps

to keep alive at least a portion of that all too quickly fading subject of Euclidean geometry. The material itself should fit the curriculum of most undergraduate courses in geometry and since, many other such relations no doubt exist, there is good potential here for student research projects. Since, in our view, there is no greater motivator than the joy of discovery, such activity is a necessary part of the teaching-learning process. As a concluding remark we note that while much has been written, so also much remains to be discovered about that most fascinating configuration—the triangle.

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THE K -TRANSFORMATION INDEX

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ABSTRACT

In this paper we introduce the K -transformation index, defined

$$\text{by : } F_s(\mu) = \int_0^\infty f(t) K_\mu(st) dt, \mu = \sigma + i\tau, s \in C \quad (1)$$

Two inversion theorems and an application of (1) on the theory of diffraction of transient electromagnetic waves are analysed.

1. INTRODUCTION

The Kontorovich-Lebedev transformation is defined by

$$F(\tau) = \int_0^\infty f(t) K_{i\tau}(t) dt, \tau > 0$$

where $K_\mu(t)$ is the modified Bessel function of third kind and order μ .

The corresponding inversion formula, under certain conditions [5], is given by :

$$f(t) = \frac{2}{\pi^2 t} \int_0^\infty \tau \sinh \pi \tau K_{i\tau}(t) F(\tau) d\tau.$$

Later N. N. Lebedev [6] define a Kontorovich-Lebedev type integral transformation of a real valued function f (I -transformation index) by :

$$H_s(\mu) = \int_0^\infty f(t) I_\mu(st) dt, \mu = \sigma + i\alpha, s \in C, \quad (1.1)$$

where $I_\mu(t)$ is the modified Bessel function f of first kind and order μ . The inversion formula is [6] :

$$f(t) = \frac{1}{\pi i t} \int_{\sigma-i\infty}^{\sigma+i\infty} \mu H_s(\mu) K_\mu(st) d\mu, \sigma > 1.$$

Now we define the K -transformation index by :

$$F_s(\mu) \int_0^\infty f(t) K_\mu(st) dt, \mu = \sigma + i\tau, s \in C. \quad (1.2)$$

The inversion method presented in [6] is utilised here to give an inversion formula for (1.2) :

$$f(t) = \frac{1}{\pi i t} \int_{\sigma-i\infty}^{\sigma+i\infty} \mu F_s(\mu) I_\mu(st) d\mu, \sigma > 1.$$

The same inversion formula is proved by using the method of Green's functions developed in [3] .

Also we study a theorem of convergence and some operational rules for (1.1). Relations between (1.2) and Laplace and iterated Laplace transformations are presented in Propositions 2.2 and 2.3 respectively .

Finally, we solve a problem of diffraction of transient electromagnetic waves by a wedge (see [2]) by using the Laplace transform and our transformation .

2. Some properties of the K -transformation index .

We now establish some properties of the K -transformation index.

Theorem 2.1 (Convergence)

Let $F_s(\mu) = \int_0^\infty f(t) K_\mu(st) dt$, where $s, \mu \in \mathbb{C}$ and $f(t)$ represents

a function locally integrable on $(0, \infty)$ such that :

$$f(t) = O(t^\beta) \text{ as } t \rightarrow 0^+ \quad (\beta \in \mathbb{R}) \quad (2.1)$$

$$\text{and } f(t) = O(e^{\lambda t}), \text{ as } t \rightarrow \infty \quad (\lambda \in \mathbb{R}, \operatorname{Re}(s) > \lambda) \quad (2.2)$$

If $-1-\beta < \operatorname{Re}(\mu) < 1+\beta$, then $F_s(\mu)$ is absolutely convergent.

Proof. In effect, let $\varepsilon > 0$, $\varepsilon < T < \infty$. We have :

$$\int_0^\infty |f(t)| |K_\mu(st)| dt = \int_0^\varepsilon |f(t)| |K_\mu(st)| dt + \int_\varepsilon^T |f(t)| |K_\mu(st)| dt + \int_T^\infty |f(t)| |K_\mu(st)| dt. \quad \text{The second}$$

of integrals exists since f is a locally integrable function on $(0, \infty)$.

$$\text{Now, } \int_0^\varepsilon |f(t)| |K_\mu(st)| dt = \int_0^\varepsilon |f(t)| |t^{-\mu} - t^\mu| dt \leq$$

$$\leq \int_0^\varepsilon |f(t)| t^{-\operatorname{Re}(\mu)} dt + \int_0^\varepsilon |f(t)| t^{\operatorname{Re}(\mu)} dt. \text{ By the hypothesis (2.1) and}$$

since $\beta > \max(-1+\operatorname{Re}(\mu), -1-\operatorname{Re}(\mu))$, we can deduce that

$$\int_0^\infty |f(t)| |K_\mu(st)| dt \text{ is convergent. Moreover, one has}$$

$$\int_T^\infty |f(t)| |K_\mu(st)| dt \leq \int_T^\infty |f(t)| t^{-1/2} e^{-\operatorname{Re}(s)t} dt.$$

Hence, in virtue of (2.2) this integral is convergent .

The function $K_\mu(st)$ verifies the modified Bessel's differential equation : $t^2 y'' + ty' - (\mu^2 + s^2 t^2) y = 0$. Therefore,

$$A_{s,t}^k (K_\mu(st)) = \mu^{2k}(st), \text{ where } A_{s,t} = t^2 D^2 + tD - s^2 t^2 .$$

Proposition 2.1

Let $f \in C^2(0, \infty)$, such that :

$$f(t) = O(t^8) \text{ as } t \longrightarrow 0^+, \quad (2.3)$$

$$f(t) = O(e^{\lambda t}) \text{ as } t \longrightarrow \infty, (\lambda \in \mathbb{R}, \operatorname{Re}(s) > \lambda)$$

If $-1-\beta < \operatorname{Re}(\mu) < 1 + \beta$:

$$\int_0^\infty A'_{s,t}(f(t)) K_\mu(st) dt = \mu^2 \int_0^\infty f(t) K_\mu(st) dt,$$

where $A'_{s,t}(f(t)) = t^2 D^2 f(t) + 3t Df(t) + (1 - s^2 t^2) f(t)$

Proof. The conditions (2.3) imply the hypothesis of Theorem 2.1 and, therefore, f is transformable. By using the rule of integration by parts we get :

$$\begin{aligned} \int_0^\infty A'_{s,t}(f(t)) K_\mu(st) dt &= \int_0^\infty f(t) A_{s,t}(K_\mu(st)) dt \\ &= \mu^2 \int_0^\infty f(t) K_\mu(st) dt. \end{aligned}$$

Next, we establish relations between the Laplace and iterated Laplace transforms and K -transformation index .

Proposition 2.2

If $f(t)$ is a Laplace transformable function, we have :

$$L\{f(t)\} = (2/\pi)^{1/2} \int_0^\infty \sqrt{st} K_{1/2}(st) f(t) dt.$$

Proof. In effect, $e^{-st} = (2st/\pi)^{1/2} K_{1/2}(st)$ and therefore

$$\int_0^\infty f(t) e^{-st} dt = (2/\pi)^{1/2} \int_0^\infty (st)^{1/2} K_{1/2}(st) f(t) dt.$$

Proposition 2.3

Let $\mu \in C$ and $Re(s^2) > 0$. Then : $\int_0^\infty f(t) K_\mu(st) dt =$

$$= \left(\frac{1}{2}\right)^{\mu+2} s^{-\mu} L \left\{ \tau^{-\mu-1} L \left\{ t^{(\mu-1)/2} f(\sqrt{t}) ; t \longrightarrow \frac{1}{4\tau} \right\} ; \tau \longrightarrow s^2 \right\}.$$

Proof : Since $\int_0^\infty f(t) K_\mu(st) dt = \left(\frac{1}{2}\right)^{\mu+1} \int_0^\infty f(t) (st)^\mu$

$$\int_0^\infty e^{-\tau - \frac{(st)^2}{4\tau}} \tau^{-\mu-1} d\tau dt.$$

By inverting to the order of integration and making $t^2 = h$,

$$\int_0^\infty f(t) K_\mu(st) dt = \left(\frac{1}{2}\right)^{\mu+2} s^\mu \int_0^\infty e^{-\tau} \tau^{-\mu-1} \int_0^\infty h^{\frac{\mu-1}{2}} e^{-\frac{s^2 h}{4\tau}} f(\sqrt{h}) dh d\tau.$$

Now, making $\tau = s^2 x$, the last expression is equal to :

$$\left(\frac{1}{2}\right)^{\mu+2} s^{-\mu} \int_0^{\infty} e^{-s^2 x} x^{-\mu-1} \int_0^{\infty} h^{\frac{\mu-1}{2}} e^{-\frac{s^2 h}{4x}} f(\sqrt{h}) dh dx =$$

$$= \left(\frac{1}{2}\right)^{\mu+2} s^{-\mu} L \left\{ \tau^{-\mu-1} L \left\{ t^{(\mu-1)/2} f(\sqrt{t}) ; t \rightarrow \frac{1}{4\tau} \right\} ; \tau \rightarrow s^2 \right\}.$$

From this relation, if $f(t) = t^{\beta}$, $\beta \in C$, $\operatorname{Re}(s^2) > 0$, $\operatorname{Re} \beta > |\operatorname{Re} \mu| - 1$,

$$\int_0^{\infty} t^{\beta} K_{\mu}(st) dt = 2^{\beta-1} \frac{1}{s^{\beta+1}} \Gamma\left(\frac{\mu+\beta+1}{2}\right) \Gamma\left(\frac{\beta-\mu+1}{2}\right)$$

3. An inversion formula.

In this section we prove an inversion formula for (1.1),

The main result of this paper is :

Theorem 3.1 (Inversion formula)

Let $F_s(\mu) = \int_0^{\infty} f(t) K_{\mu}(st) dt$, $\mu = \sigma + i\tau$, $\sigma > 1$, where

$f(t)$ is bounded variation in each interval (a, A) , $0 < a < A$. (3.1)

and $f(t) K_{\mu}(st) \in L(0, \infty)$.

Then
$$f(t) = \frac{1}{\pi i t} \int_{\sigma-i\infty}^{\sigma+i\infty} \mu F_s(\mu) I_{\mu}(st) d\mu$$

Proof. Let $J(t, T) = \frac{1}{\pi i t} \int_{\sigma-i\infty}^{\sigma+i\infty} \mu F_s(\mu) I_{\mu}(st) d\mu$.

We will prove that

$$\lim_{T \rightarrow \infty} J(t, T) = f(t).$$

Since
$$J(t, T) = \int_0^\infty f(\zeta) d\zeta \frac{1}{\pi i t} \int_{\sigma-iT}^{\sigma+iT} \mu K_\mu(s\zeta) I_\mu(st) d\mu$$

and making the change of variables $\zeta = te^\theta$

$$J(t, T) = \int_{-\infty}^\infty f(te^\theta) e^\theta G_T(\theta) d\theta$$

where
$$G_T(t) = \frac{1}{\pi i} \int_{\sigma-iT}^{\sigma+iT} \mu I_\mu(st) K_\mu(st e^\theta) d\mu.$$

Dixon and Ferrar [4] gave the integral representation :

$$K_\mu(st e^\theta) I_\mu(st) = \frac{1}{2} \int_0^\infty e^{-\mu\eta} J_0(st e^{\theta/2} \Psi) d\eta, \quad (3.2)$$

where $\Psi = (2ch\eta - 2ch\theta)^{1/2}$, $-\infty < \theta < \infty$, and $\text{Re}\mu > 1/4$. Here J_0 is the Bessel function of order zero.

By using (3.2) and integrating by parts

$$G_T(\theta) = \frac{e^{-\sigma\theta} \text{sen} T\theta}{\pi \theta} + R_T(\theta), \text{ where}$$

$$R_T(\theta) = -\frac{1}{\pi} \int_0^\infty e^{-\sigma\eta} \frac{\text{sen} \eta T}{\eta} \frac{\partial}{\partial \eta} J_0(st e^{\theta/2} \Psi) d\eta.$$

Now, by Theorem of Dirichlet [1, p. 383]

$$\lim_{T \rightarrow \infty} \int_0^\infty f(te^\theta) e^{-\sigma\theta} \frac{\text{sen} T\theta}{\theta} d\theta = f(t)$$

We have to see that

$$\lim_{T \rightarrow \infty} \int_{-\infty}^{\infty} f(te^{\theta}) R_T(\theta) d\theta = 0.$$

Following once again some manipulations of Lebedev [6],

one has

$$R_T(\theta) = \frac{-ste^{\theta/2}}{\pi T} \left\{ \frac{1}{2} st e^{-(\alpha + \frac{1}{2})\theta} \frac{se^{\eta h\theta}}{\theta} \cos T\theta + \right.$$

$$+ \int_0^{\infty} e^{-\sigma\eta} \cosh\eta \cos T\eta \left[\left(\frac{1}{\eta} - \frac{th\eta}{\eta^2} - \sigma \frac{th\eta}{\eta} \right) \frac{J_1(ste^{\theta/2}\Psi)}{\Psi} - \right.$$

$$\left. \left. - st e^{\theta/2} \frac{th\eta \sinh\eta}{\eta \Psi^2} J_2(ste^{\theta/2}\Psi) \right] d\eta \right\}.$$

Moreover, by virtue of well-known boundedness properties, we can prove for $\theta \geq 0$

$$|R_T|_{\theta \geq 0} \leq \frac{A_1 |s|^2 t^2 e^{-\sigma\theta}}{T} +$$

$$+ \frac{|s| t e^{\theta/2}}{T} \int_0^{\infty} e^{-(\sigma-1)\eta} \left[A_1 |s| t e^{\theta/2} + A_3 |s| t e^{\theta/2} + A_4 |s|^3 \right.$$

$$\left. t^3 e^{\theta/2} \right] d\eta$$

$$= \left[\frac{A_1 |s|^2 t^2}{T} + \frac{A_5 |s|^2 t^2 e^{\frac{3}{2}\theta}}{T} + \frac{A_{\sigma} |s| t^4 e^{\frac{3}{2}\theta}}{T} \right] e^{-\sigma\theta},$$

being $A_i, i=1, \dots, \sigma$ constants, and provided that $\sigma > 1$.

On the other hand, for $\theta \leq 0$

$R_{T/\theta \leq 0} = R_1 + R_2$, where

$$R_1 = \frac{-ste^{\theta/2}}{\pi T} \left\{ \frac{1}{2} st e^{-(\sigma + \frac{1}{2})\theta} \frac{\sinh \theta}{\theta} \cos T\theta + \right. \\ \left. + \int_{-\theta}^{\infty} e^{-\sigma \eta} \cosh \eta \left[\left(\frac{1}{\eta} - \frac{th\eta}{\eta^2} - \sigma \frac{th\eta}{\eta} \right) \frac{J_1(ste^{\theta/2} \Psi)}{\Psi} - \right. \right. \\ \left. \left. - st e^{\theta/2} \frac{th\eta}{\eta} \frac{\sinh \eta}{\Psi^2} J_2(ste^{\theta/2} \Psi) \right] \right.$$

and

$$R_2 = \frac{-ste^{\theta/2}}{\pi T} \int_{\theta}^{-\theta} e^{-\sigma \eta} \cosh \eta \cos T\eta \left\{ \left(\frac{1}{\eta} - \frac{th\eta}{\eta^2} - \sigma \frac{th\eta}{\eta} \right) \frac{I_1(ste^{\theta/2} \Psi_1)}{\Psi_1} \right. \\ \left. - st e^{\theta/2} \frac{th\eta}{\eta} \frac{\sinh \eta}{\Psi_1^2} I_2(ste^{\theta/2} \Psi_1) \right\} d\eta$$

being $\Psi = i\Psi_1$, and $\Psi_1 = \left(2\cosh \theta - 2\cosh \eta \right)^{1/2}$.

In effect, by using boundedness properties of the Bessel functions and integrating, we get

$$|R_1| \leq \frac{B_1 |s|^2 t^2 e^{-\sigma \theta}}{T} + \frac{|s| t e^{\theta/2}}{T} \int_{-\theta}^{\infty} e^{-(\sigma-1)\eta} \\ \left[B_2 |s| t e^{\theta/2} + B_3 |s| t e^{\theta/2} + B_4 |s|^3 t^3 e^{\theta/2} \right] d\eta =$$

$$= \frac{B_1 |s|^2 t^2 e^{-\sigma\theta}}{T} + \frac{B_5 |s|^2 t^2 e^{\sigma\theta}}{T} + \frac{B_6 |s|^4 t^4 e^{\sigma\theta}}{T} -$$

provided that $\sigma > 1$, and where $B_i, i=1, \dots, 6$, are constants.

Also,

$$|R_2| \leq \frac{A_7 |s| |te^{\theta/2}|}{T} \int_0^{-\theta} e^{(\sigma+1)\eta} \frac{|I_1(ste^{\theta/2} \Psi_1)|}{\Psi_1} d\eta +$$

$$+ \frac{A_8 |s|^2 t^2 e^{\theta}}{T} \int_0^{-\theta} e^{(\sigma+1)\eta} \frac{\sinh \eta}{\Psi_1^2} |I_2(ste^{\theta/2} \Psi_1)| d\eta = Q_1 + Q_2$$

where

$$Q_1 = \frac{A_7 |s| |te^{\theta/2}|}{T} \int_0^{\varepsilon} e^{(\sigma+1)\eta} \frac{|I_1(ste^{\theta/2} \Psi_1)|}{\Psi_1} d\eta +$$

$$+ \frac{A_7 |s| |te^{\theta/2}|}{T} \int_{\varepsilon}^{\theta} e^{(\sigma+1)\eta} \frac{|I_1(ste^{\theta/2} \Psi_1)|}{\Psi_1} d\eta$$

and

$$Q_2 = \frac{A_8 |s| |te^{\theta/2}|}{T} \left[\frac{|I_1(2ste^{\theta/2} \sinh \theta/2)|}{2 \sinh \theta/2} - \frac{1}{2} |s| |te^{(\sigma+3/2)\theta}| \right] + A_9 Q_1$$

being A_7, A_8 and A_9 suitable positive constants. Moreover, the following estimates hold :

$$\frac{A_7 |s| |te^{\theta/2}|}{T} \int_0^{\varepsilon} e^{(\sigma+1)\eta} \frac{|I_1(ste^{\theta/2} \Psi_1)|}{\Psi_1} d\eta$$

$$\leq \frac{A_7 |s| |te^{\theta/2}| |I_1(2ste^{\theta/2} \sinh \theta/2)|}{T(\sigma+1) 2 \sinh \theta/2} e^{(\sigma+1)\varepsilon-1}$$

$$\frac{A_7 |s| t e^{\theta/2}}{T} \int_{\varepsilon}^{-\theta} e^{(\sigma+1)\eta} \frac{|I_1(st e^{\theta/2} \Psi_1)|}{\Psi_1} d\eta \leq$$

$$\leq \frac{A_{10} |s| t}{T} \cosh^{\sigma+1} \theta \int_0^{\pi/2} \cos^{2\sigma+1} x |I_1(st (2e^{\theta} \cosh \theta) \sin x)| dx,$$

($A_{10} = \text{constant}$) and since

$$\int_0^{\pi/2} \cos^{2\sigma+1} x I_1(z \sin x) dx = \frac{1}{2} \left[\frac{I_{\sigma}(z) \Gamma(z+1)}{(z/2)^{\sigma}} - 1 \right], \text{ therefore}$$

$$\frac{A_7 |s| t e^{\theta/2}}{T} \int_{\varepsilon}^{-\theta} e^{(\sigma+1)\eta} \frac{|I_1(st e^{\theta/2} \Psi_1)|}{\Psi_1} d\eta \leq$$

$$\leq \frac{A_{11} \cosh^{\sigma+1} \theta}{T |st|^{\sigma} (2e^{\theta} \cosh \theta)^{\frac{\sigma+1}{2}}} \left[|I_{\sigma}(st (2e^{\theta} \cosh \theta))| - \right.$$

$$\left. \frac{|s|^{\sigma} (2e^{\theta} \cosh \theta)^{\sigma/2}}{2^{\sigma} \Gamma(\sigma+1)} \right]$$

where A_{11} is a constant. Therefore, we can obtain

$$|R_T|_{\theta} \leq \frac{M(t, \theta)}{T}$$

where $M(t, \theta)$ is a continuous function of two variables t, θ , and hence

$$\lim_{T \rightarrow \infty} \int_0^{\infty} f(te^{\theta}) e^{\theta} R_T(\theta) d\theta = 0$$

4. Inversion by Green's functions,

The modified Bessel's differential equation is

$$t^2 y'' + t y' - (s^2 t^2 + \mu^2) y = 0 \quad (4.1)$$

The delta function $\delta(t - \rho)$ can be written as

$$t \delta(t - \rho) = - \frac{1}{2\pi i} \oint G(t, \rho, \lambda) d\lambda \quad (4.2)$$

where $G(t, \rho, \lambda)$ is the Green's function of the operator in (4.1), satisfying the equation

$$\left[D^2 + \frac{1}{t} D - \left(s^2 + \frac{\mu^2}{t^2} \right) \right] G = - \delta(t - \rho)$$

and the integration is over a suitable contour in the λ -plane including all the singularities of the Green's function in terms of λ .

If $\text{Re}(\mu) > 0$ and since $I_\mu(st)$ and $K_\mu(st)$ are independent solutions of (4.1) verifying

$$R_\mu(st) \rightarrow 0, \quad t \rightarrow \infty,$$

$I_\mu(st)$ is finite as $t \rightarrow 0$, then the Green's

function $G(t, \rho, \lambda)$ can be represented as

$$G(t, \rho, \lambda) = \begin{cases} \frac{I_\mu(st) K_\mu(s\rho)}{Wt}, & t < \rho \\ \frac{I_\mu(s\rho) K_\mu(st)}{Wt}, & t > \rho \end{cases} \quad (4.3)$$

where W is the wronskian $W(I_\mu, K_\mu)$ and is a function of μ and t . The form of the two expressions of (4.3) shows that the same path of integration in (4.2) holds for $t > \rho$ and $\rho > t$. Now, the function $f(t)$, defined on $0 < t < \infty$ can be written as

$$f(t) = \int_0^{\infty} f(\rho) \delta(t-\rho) d\rho$$

Therefore by (4.3) and (4.2) we get

$$f(t) = -\frac{1}{2\pi i} \int_0^{\infty} f(\rho) \left\{ \oint \frac{K_{\mu}(s\rho) I_{\mu}(st)}{W t^2} d\lambda \right\} d\rho$$

since

$$\oint \frac{K_{\mu}(s\rho) I_{\mu}(st) - I_{\mu}(s\rho) K_{\mu}(st)}{W t} d\lambda = 0,$$

for want of singularity. Interchanging the orders of integration (assuming it to be possible) and being $\lambda = \mu^2$

$$f(t) = -\frac{1}{2\pi i} \oint 2\mu \frac{I_{\mu}(st)}{W t^2} d\mu \int_0^{\infty} f(\rho) K_{\mu}(s\rho) d\rho.$$

Using the formula $W(I_{\mu}, K_{\mu}) = -1/t$,

$$f(t) = \frac{1}{\pi i t} \oint \mu I_{\mu}(st) d\mu \int_0^{\infty} f(\rho) K_{\mu}(s\rho) d\rho$$

where the path of integration in some abscissae $\sigma > 0$ in the plane of the variable $\mu = \sigma + i\tau$, since the singularities of I_{μ} are at $\mu = 0, -1, -2, \dots$, therefore

$$f(t) = \frac{1}{\pi i t} \int_{\sigma-i\infty}^{\sigma+i\infty} \mu I_{\mu}(st) d\mu \int_0^{\infty} f(\rho) K_{\mu}(s\rho) d\rho$$

5. A Parseval relation for the K -transformation index.

Let
$$F_s(\mu) = \int_0^{\infty} f(t) K_{\mu}(st) dt$$

$$G_s(\mu) = \int_0^\infty g(t) K_\mu(st) dt,$$

the following Parseval relation holds :

$$\frac{i}{\pi^2 t} \int_{\sigma-i\infty}^{\sigma+i\infty} \mu \operatorname{sen} \mu \pi F_s(\mu) G_s(\mu) d\mu = \int_0^\infty f(t) G(t) dt$$

In effect,

$$\begin{aligned} & \frac{i}{\pi^2 t} \int_{\sigma-i\infty}^{\sigma+i\infty} \mu \operatorname{sen} \mu \pi F_s(\mu) G_s(\mu) d\mu = \\ &= \int_0^\infty g(t) \frac{i}{\pi^2 t} \int_{\sigma-i\infty}^{\sigma+i\infty} \mu \operatorname{sen} \mu \pi F_s(\mu) \frac{\pi}{2 \operatorname{sen} \mu \pi} [I_{-\mu}(st) - I_\mu(st)] d\mu dt \\ &= \int_0^\infty g(t) \frac{i}{2\pi t} \int_{\sigma-i\infty}^{\sigma+i\infty} \mu F_s(\mu) [I_{-\mu}(st) - I_\mu(st)] d\mu dt \end{aligned}$$

Now

$$\int_{\sigma-i\infty}^{\sigma+i\infty} \mu F_s(\mu) I_{-\mu}(st) d\mu = - \int_{\sigma-i\infty}^{\sigma+i\infty} \mu F_s(\mu) I_\mu(st) d\mu,$$

since $F_s(\mu) = F_s(-\mu)$.

$$\begin{aligned} & \frac{i}{\pi^2 t} \int_{\sigma-i\infty}^{\sigma+i\infty} \mu \operatorname{sen} \mu \pi F_s(\mu) G_s(\mu) d\mu = \\ &= \int_0^\infty g(t) \left(\frac{-i}{\pi t} \right) \int_{\sigma-i\infty}^{\sigma+i\infty} \mu F_s(\mu) I_\mu(st) d\mu dt = \\ &= \int_0^\infty g(t) \frac{1}{\pi i t} \int_{\sigma-i\infty}^{\sigma+i\infty} \mu F_s(\mu) I_\mu(st) d\mu dt = \int_0^\infty f(t) g(t) dt. \end{aligned}$$

6. An operational calculus.

Let P be any polynomial. Consider the differential equation

$$P (A'_{s,t}) f = g \quad (6.1)$$

where g is a given transformable function. The K -transformation index generates an operational calculus through which a solution u for (6.1) can be found. By applying this transform to (6.1) and invoking the Proposition 2.1 we obtain

$$F (\mu) = \frac{G (\mu)}{P (\mu^2)}$$

We may apply the inversion formula to determine f , solution to (6.1). The extension of this technique to simultaneous differential equations of the form (6.1) is straightforward.

7. An application of the K -transformation index.

The problem to which the K -transformation index is applied is the one considered in [2].

An infinitely long line source of uniformly distributed current, whose density is a function of time alone, lies parallel to the edge of a wedge with perfectly conducting walls in a region of conductivity σ , dielectric constant ϵ and permeability λ . Cylindrical coordinates (r, θ, z) with the tip of the wedge coincident with the z -axis are used. The angle of the wedge is taken to be α and the line source passes through (r_0, θ_0) , $0 \leq \theta_0 \leq \alpha$. Since the problem is two dimensional. Maxwell's equations (in m. k. s. units) for the region within the wedge are

$$\begin{aligned}
 a) \quad & \frac{1}{r} \frac{\partial}{\partial r} (r H_\theta) - \frac{1}{r} \frac{\partial H_r}{\partial \theta} = (\sigma + \varepsilon \frac{\partial}{\partial t}) E_z + M(t) \frac{\delta(r-r_0)}{r} \delta(\theta-\theta_0) \\
 b) \quad & \frac{1}{r} \frac{\partial E_z}{\partial \theta} = -\lambda \frac{\partial H_r}{\partial t}, \quad c) \quad \frac{\partial E_z}{\partial r} = \lambda \frac{\partial H_\theta}{\partial t}
 \end{aligned} \tag{7.1}$$

where $r^{-1} M(t) \delta(r-r_0) \delta(\theta-\theta_0)$ represents a current density corresponding to a line source passing through (r_0, θ_0) . $M(t)$ is a function of time alone and $\delta(x)$ denotes the Dirac delta function. The conditions on the boundary are

$$E_z = 0 \text{ on } \theta = 0, \alpha \tag{7.2}$$

$$H_r = H_\theta = E_z = 0 \quad \text{at } t = 0 \text{ and as } r \rightarrow \infty$$

A Laplace transform in t and the K -transformation index in r , with $s^2 = \mu\eta(\sigma + \varepsilon\eta)$, are used to reduce these equations to an ordinary differential equation in θ .

Multiplying equation 7.1a) by $r K_\mu(sr) e^{-\eta t}$, 7.1b) by $K_\mu(sr) e^{-\eta t}$, 7.1c) by $r \frac{\partial}{\partial r} K_\mu(sr) e^{-\eta t}$ and integrating with respect to t and r over the range $(0, \infty)$ we find

$$\begin{aligned}
 -H_\theta - \frac{dH_r}{d\theta} &= (\sigma + \varepsilon\eta) \xi_z^{(1)} + \bar{M}(\eta) K_\mu(sr_0) \delta(\theta - \theta_0) \\
 \frac{d\xi_z}{d\theta} &= -\lambda\eta H_r
 \end{aligned} \tag{7.3}$$

$$-s^2 \xi_z^{(1)} - \lambda^2 \xi_z = \lambda\eta H_\theta$$

where

$$\xi_z(\mu, \theta, \eta) = \int_0^\infty \frac{\bar{E}_z(r, \theta, \eta)}{r} K_\mu(sr) dr =$$

$$= \int_0^\infty \frac{K_\mu(sr)}{r} dr \int_0^\infty E_z(r, \theta, t) e^{-\eta t} dt$$

$$\xi_z^{(1)}(\mu, \theta, \eta) = \int_0^\infty \bar{E}_z K_\mu(sr) r dr,$$

$$H_\theta(\mu, \theta, \eta) = \int_0^\infty \bar{H}_\theta r \frac{\partial K_\mu(sr)}{\partial r} dr$$

$$H_r(\mu, \theta, \eta) = \int_0^\infty \bar{H}_r K_\mu(sr) dr$$

and the bar above a component denotes its Laplace transform with respect to t .

From equations (7.3) we find that the equation for ξ_z is

$$\left(\frac{d^2}{d\theta^2} + \mu^2 \right) \xi_z = \lambda \eta \bar{M}(\eta) K_\mu(sr_0) \delta(\theta - \theta_0) \quad (7.4)$$

The solution of equation (7.4) subject to the boundary conditions (7.2) can be written as

$$\xi_z = \lambda \eta \bar{M}(\eta) \frac{\text{sen} \mu(\alpha - \theta_0) \text{sen} \mu \theta}{\mu \text{sen} \mu \alpha} K_\mu(sr_0), \quad 0 \leq \theta < \theta_0$$

When $\theta > \theta_0$ the positions of θ and θ_0 are interchanged. Inverting with respect to the K -transformation index we have ($\theta < \theta_0$)

$$E_z (r, \theta, t) = \frac{\lambda}{2\pi^2} \int_{c-i\infty}^{c+i\infty} \eta \bar{M}(\eta) e^{\eta t} d\eta.$$

$$\int_{\sigma-i\infty}^{\sigma+i\infty} \frac{\text{sen}\mu (a-\theta_0) \text{sen}\mu\theta}{\text{sen}\mu a} K_\mu(sr_0) I_\mu(sr) d\mu$$

Final Observation .

We also note that the K -transformation index may be extended to distributions along the lines of A. H. Zemanian [12]. This will require the construction of a Fréchet space that contains the kernel $K_\mu(st)$ as an element . The transform will then be defined on the dual space as the application of the distribution to this kernel . This will be the subject of a later paper .

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APPLICATIONS OF FIXED POINTS TO APPROXIMATION THEORY

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Several interesting results have been given in Approximation theory using the fixed point theory . A very good source of such illustrations may be found in a paper due to Chener [2] . For a detailed survey paper one is referred to [5] .

The work of Brosowski [1], Singh [6], and Hicks and Humphries [4] involves nonexpansive mappings . Recall that if $f : X \rightarrow X$, where X is a normed linear space, and $\|fx - fy\| \leq \|x - y\|$ for $x, y \in X$, then f is said to be nonexpansive map

The aim of this paper is to prove results in this direction and derive known results as corollaries .

We recall that, given $x \in X$ and $C \subset X$, the set of the best C -approximants x consist of all $y \in C$ such that $\|y-x\| = \inf \{ \|z-x\|, z \in C \}$.

The main theorem is given below .

Theorem 1. Let C be a subset of a Banach space X and $T : X \rightarrow X$ be a mapping. Let $T(\partial C) \subset C$ (∂C stands for the boundary of C), $Tx_0 = x_0$, $x_0 \in X$ and T satisfy the following condition :

$$\|Tx - Ty\| \leq \alpha \|x - y\| + \beta (\|x - Tx\| + \|y - Ty\|) + \gamma (\|x - Ty\| +$$

$\|y - Tx\|)$ for all $x, y \in X$, where α, β and γ are reals with $\alpha + 2\beta + 2\gamma \leq 1$.

Further, if the set D : the set of best C -approximants to x_0 , is non-empty compact convex and T is continuous on D , then T has a fixed point in D closest to x_0 .

Remarks : In [1] a similar result is proved for a linear, nonexpansive map T from C to itself, and an extended version in [6] where $T: C \rightarrow C$ is taken. Here we do not require linearity, nonexpansiveness of T on X and $T: C \rightarrow C$ not needed, only $T(\partial C) \subset C$ serves the purpose.

Proof : First we show that

$$\|Tx - Tx_0\| \leq \|x - x_0\| \text{ for all } x \in X.$$

Clearly,

$$\|Tx - Tx_0\| \leq \alpha \|x - x_0\| + \beta (\|x - Tx\| + \|x_0 - Tx_0\|) + \gamma (\|x - Tx_0\| + \|x_0 - Tx\|).$$

Since $Tx_0 = x_0$, We get

$$\|Tx - Tx_0\| \leq \alpha \|x - x_0\| + \beta \|x - Tx\| + \gamma (\|x - x_0\| + \|Tx_0 - Tx\|).$$

Now, $\|x - Tx\| \leq \|x - x_0\| + \|Tx - Tx_0\|$, therefore we have

$$\|Tx - Tx_0\| \leq \frac{\alpha + \beta + \gamma}{1 - \beta - \gamma} \|x - x_0\|,$$

$$\text{i. e., } \|Tx - Tx_0\| \leq \|x - x_0\| \text{ for all } x \in X \quad (1).$$

We claim, that $T: D \rightarrow D$.

Let $y \in D$. Then

$$\|Ty - x_0\| = \|Ty - Tx_0\| \leq \|y - x_0\| \text{ by (1).}$$

Clearly y cannot be in C , hence $y \in \partial C$ (see e. g. [4], Lemma 1).

Since y is nearest to x_0 and $T(\partial C) \subset C$, so Ty is also nearest to x_0 , i. e. $Ty \in D$.

Since D is a non-empty compact convex subset of X and $T : D \rightarrow D$ is a continuous map therefore T has a fixed point in D by Schauder fixed point theorem. So $Tx = x$ say, where $x \in D$ closest to x_0 .

Theorem 2. Let $T : X \rightarrow X$ be a mapping on Banach space X satisfying the following :

- I) $Tx_0 = x_0, x_0 \in X$,
- II) $T(\partial C) \subset C$, where C is a subset of X ,
- III) the set D : the set of best C -approximants to x_0 is compact and starshaped,
- IV) T is nonexpansive on D and $\|Tx - Tx_0\| \leq \|x - x_0\|$ for all $x \in D$. Then T has a fixed point in D closest to x_0 .

Recall that a set B is said to be starshaped with star center $p \in B$ if

$\lambda x + (1 - \lambda)p \in B, 0 \leq \lambda \leq 1$ for all $x \in B$. A convex set is always starshaped, but not conversely.

Singh [7] proved a similar theorem under the condition $T : C \rightarrow C$. It is shown in [4] that $T(\partial C) \subset C$ suffices.

Our proof follows similar lines as in [6].

Proof: As in Theorem 1 it is easy to show that $T : D \rightarrow D$.

Since T is a nonexpansive mapping on D and D is starshaped with star center p therefore we define $Tk_n x = k_n Tx + (1 - k_n)p$ for all $x \in D$, where $\{k_n\}$ is a sequence of real numbers, $0 \leq k_n < 1$, converges

to 1 as $n \rightarrow \infty$. Now, $T_n : D \rightarrow D$ for each n , since so is T (We write n in place of k_n).

Also, $\|T_n x - T_n y\| = k_n \|Tx - Ty\| \leq k_n \|x - y\|$. Since T is non expansive on D and $0 \leq k_n < 1$, then

$$\|T_n x - T_n y\| < \|x - y\|, \quad x \neq y \text{ for all } x, y \in D.$$

Then each T_n has a unique fixed point say y_n , for $n = 1, 2, \dots$ (see Edelstein [3]), i.e.

$$T_n y_n = y_n, \quad n = 1, 2, 3, \dots$$

Since D is compact therefore $\{y_n\}$ has a subsequence $\{y_{n_i}\}$ converging to y say.

We claim that $Ty = y$ in D .

Since

$$y_{n_i} = T_{n_i} y_{n_i} = (1 - k_{n_i})p + k_{n_i} Ty_{n_i},$$

taking limit as $i \rightarrow \infty$ and using the fact that T is continuous we get that

$$y = Ty \text{ since } k_{n_i} \rightarrow 1.$$

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A REMARK ON A SMOOTHED FOURIER SERIES *

By

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Many recent papers studied various problems connected to the approximation of a given function f , defined on the N -dimensional torus T^N , by its Fourier polynomials, smoothed by multiplying every coefficient $f(n)$ by a factor of the form $[1 + \sigma P(n)]^{-1}$, where P is a homogeneous polynomial of even degree K and σ is a positive real parameter. See [1], [2], [4], [5].

In particular if $f \in C(T^N)$, (the space of continuous functions with the supremum norm) it is known ([1] Th. 1) that

$$f_{\sigma}(t) \sim \sum_{n \in \mathbb{Z}^N} \frac{f(n)}{1 + \sigma P(n)} e^{2\pi i n t}, \quad t \in T^N$$

($\mathbb{Z}^N$ the lattice of integer points of R^N), is a function of $C(T^N)$ and

$$\|f - f_{\sigma}\|_{\infty} \rightarrow 0 \quad \text{when } \sigma \rightarrow 0^+$$

Moreover if f belongs to the Lipschitz $\text{Lip}(\alpha, C(T^N))$, $0 < \alpha \leq 1$, and c is a suitable positive constant ([2], Th. 4)

$$\|f - f_{\sigma}\|_{\infty} \leq c \sigma^{\alpha/k}$$

and, if N is odd, the above approximation is sharp, at least when P is

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radial [5].

In this paper we show that a similar result still holds when $N=2$ and $K=4$. Indeed we prove the following

Theorem . Let $N=2$ and P the radial polynomial of degree $K=4$, i. e $P(x) = |x|^4$.

Then there exists $M > 0$ such that

$$\sup_{f \in \text{Lip}(\alpha, C(T^2))} \|f - f_\sigma\|_\infty > M \sigma^{\alpha/k}$$

Remark . This theorem and the one proved in [5] guarantee that the given approximation is the best one both in every odd dimension N and in a particular case when $N=2$. These results suggest that the same holds in every dimension, although it seems that the demonstrative techniques adopted can not be extended to the general case .

Proof : From [1] Th. 5 we know that given $g : \mathbb{R}^N \rightarrow \mathbb{R}$ defined by $g(x) = [1 + P(x)]^{-1}$, if we call G the function belonging to $L_1(\mathbb{R}^N)$ such that $G(x) = g(x) \forall x \in \mathbb{R}^N$ and define :

$$G_\sigma(x) = \sigma^{-N/K} G(x\sigma^{-1/k})$$

$$\text{and} \quad K_\sigma(t) = \sum_{n \in \mathbb{Z}^N} G_\sigma(t+n) \quad , t \in T^N$$

then we have

$$K_\sigma(t) \sim \sum_{n \in \mathbb{Z}^N} \frac{1}{1 + \sigma P(n)} e^{2\pi i n t} \quad \text{and} \quad f_\sigma = f * K_\sigma.$$

We also recall that, by hypothesis, $g \in L_1(\mathbb{R}^2)$ and so, by the inversion formula, $g(x) = G(x)$. Being g radial also G is and it can be computed, ([6] p. 155) by the Hankel's transform

$$G(x) = G(|x|) = 2\pi \int_0^{+\infty} g(|x|) J_0(2\pi |x| |y|) |x| d|x|.$$

By [3] Vol. 11, p. 7 we get

$$G(x) = -2\pi \operatorname{Kei}_0(2\pi |x|)$$

where Kei_0 is one of the Kelvin functions, related to the modified Bessel function K_ν by

$$\operatorname{Kei}_0(x) = (2i)^{-1} [K_0(x e^{i\pi/4}) - K_0(x e^{-i\pi/4})].$$

For every $\alpha \in (0, 1]$ let us define

$$f_\alpha(x) = \begin{cases} |x|^\alpha & |x| \leq 1/2 \\ (1 - |x|)^\alpha & 1/2 < |x| < 1 \end{cases}$$

The function f_α restricted to the fundamental domain

$$Q_2 = \{x \in \mathbb{R}^2 : -1/2 \leq x_j < 1/2, j = 1, 2\}$$

defines a continuous function; furthermore $f_\alpha \in \operatorname{Lip}(\alpha, C(T^2))$,

Define $f_{\alpha, \sigma} = K_\sigma * f_\alpha$ and continue f_α periodically in each variable throughout the whole space $\mathbb{R}^2$. Then

$$\begin{aligned} \|f_{\alpha, \sigma} - f_\alpha\|_\infty &\geq |f_{\alpha, \sigma}(0)| = \\ &= \left| \int_{Q_2} K_\sigma(-x) f_\alpha(x) dx \right| = \left| \sum_{n \in \mathbb{Z}^2} \int_{Q_2 + n} G_\sigma(x+n) f_\alpha(x) dx \right| = \\ &= \left| \int_{\mathbb{R}^2} G_\sigma(x) f_\alpha(x) dx \right| = \left| \int_{\mathbb{R}^2} G(x) f_\alpha(\sigma^{1/k} x) dx \right|. \end{aligned}$$

If we consider g defined as above for some positive constant c we have

$$\begin{aligned} \|f_{\alpha, \sigma} - f_{\alpha}\|_{\infty} &\geq c \cdot \left| \int_0^{+\infty} G(|x|) f_{\alpha}(\sigma^{1/k} |x|) |x| d|x| \right| \\ &\geq c \sigma^{\alpha/k} \left| \int_0^{1/2 \sigma^{-1/k}} G(|x|) |x|^{1+\alpha} d|x| \right| - c \left| \int_{1/2 \sigma^{-1/k}}^{+\infty} |x| G(|x|) \right. \\ &\quad \left. f_{\alpha}(\sigma^{1/k} |x|) d|x| \right| = c \sigma^{\alpha/k} I_1(\sigma) - c I_2(\sigma). \end{aligned}$$

Let us name :

$$I = \int_0^{+\infty} |x|^{1+\alpha} G(|x|) d|x|.$$

From [3, Vol. 1, p 391], we get

$$I = 1/2 \pi^{-(1+\alpha)} \Gamma^2(1+\alpha/2) \cos(\alpha\pi/4).$$

Since I is positive for every $\alpha \in (0, 1]$, we can choose δ sufficiently small such that there exists $\sigma_0 = \sigma_0(\delta)$ in order to have

$$I_1(\sigma) > \delta \quad \forall \sigma < \sigma_0(\delta).$$

On the other hand recalling that

$$K_1(y) \sim \left(\frac{\pi}{2y} \right)^{1/2} e^{-y} \quad \text{when } y \rightarrow \infty$$

we can say that

$$I_2(\sigma) < c \sigma^{-1/2k} e^{-3/2 \sigma^{-1/k}} = o(\sigma^{\lambda}) \quad \forall \lambda > 0 \text{ when } \sigma \rightarrow 0^+$$

The proof is so complete.

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SLIP FLOW IN A ROTATING CHANNEL

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ABSTRACT

An attempt has been made to study the effects of the slip parameter on the flow formed when a straight channel bounded by two parallel walls which liquid is flowing under a constant pressure gradient is rotated about an axis perpendicular to the walls. A method for setting up an experiment to test the theoretical conclusions of the paper has been suggested.

1. INTRODUCTION

Greenspan [1] presented a detailed account of motion of rotating liquids enclosed within a body or vice-versa, including his own general theory of a contained rotating liquid. Rott and Lewellen [2] discussed the problems relating to the boundary layers and their interactions in rotating flows, and gave many examples of such interactions. Another example of such interaction is the flow of a liquid under a constant pressure gradient between two rotating parallel walls, considered by Vidyanidhi and Nigam [6]. Accounting for the first order slip as in the works of Soundalgekar [4], and Sastry and Bhadram [3], this latter problem has been dealt with here.

2. BASIC EQUATIONS AND SOLUTION

The equations of motion and continuity for the steady state in

a rotating frame of reference $O'X'Y'Z'$ as given by Squire [5] for an incompressible liquid are

$$\rho (\vec{u}' \cdot \nabla') \vec{u}' - 2\rho (\vec{u}' \times \vec{\Omega}') = -\nabla' \pi' + \rho \nu \nabla'^2 \vec{u}' \quad (1)$$

$$\nabla' \cdot \vec{u}' = 0 \quad (2)$$

where

$$\pi' = P' - \frac{1}{2} \rho |\vec{\Omega}' \times \vec{r}'|^2 \text{ and } \vec{u}', \vec{\Omega}'$$

and $\vec{r}'$ are the velocity vector, angular velocity, and the position vector, respectively.

We choose a right-handed Cartesian system such that the Z' -axis is perpendicular to the motion of the liquid flowing under the action of a constant pressure gradient $-\frac{\partial \pi'}{\partial x'}$ in the direction of X' -axis between two parallel walls $Z' = \pm L$ (Stationary relative to $O'X'Y'Z'$).

Assuming

$$\vec{u}' = \left\{ u'_{x'}(z'), u'_{y'}(z'), 0 \right\} \quad (3)$$

$$\vec{\Omega}' = (0, 0, \Omega') \quad (4)$$

and introducing non-dimensional quantities

$$r' = rL, u' = \frac{uQ}{\rho L}, \quad (5)$$

$$\Omega' = \frac{\Omega Q}{\rho L^2}, \quad \pi' = \frac{\pi Q^2}{\rho L^2}$$

where Q is the mass flux given by

$$Q = (Q_{x'}'^2 + Q_{y'}'^2)^{\frac{1}{2}}$$

$$Q_{x'}' = \int_{-L}^L \rho u'_{x'} dz', \quad Q_{y'}' = \int_{-L}^L \rho u'_{y'} dz',$$

the equation (1) reduces to

$$-2u_y = P + \frac{1}{\alpha^2} \frac{d^2 u_x}{dz^2} \quad (6)$$

$$2u_x = \frac{1}{\alpha^2} \frac{d^2 u_y}{dz^2} \quad (7)$$

where

$$\alpha^2 = \frac{\Omega'}{\nu} L^2$$

$$P = - \frac{L}{Q\Omega'} \frac{\partial \pi'}{\partial x'} \quad (8)$$

We seek the solution of (6) and (7) subject to the boundary conditions

$$\left. \begin{aligned} u_x &= \beta \frac{\partial u_x}{\partial z} \\ u_y &= \beta \frac{\partial u_y}{\partial z} \end{aligned} \right\} \text{ at } z = \pm 1 \quad (9)$$

where β denotes non-dimensional first-order velocity slip as in the works of Soundalgekar [4], and Sastry Bhadram [3].

The solution of the above boundary-value problem is given by

$$\begin{aligned} \frac{u_z}{P_0} = & [\alpha \beta (\cosh (\alpha) \cos (\alpha) \cos (\alpha z) \sinh (\alpha z) \\ & + \sin (\alpha) \cosh (\alpha) \sin (\alpha z) \sinh (\alpha z) \\ & + \cosh (\alpha z) \sin (\alpha z) \cos (\alpha) \cosh (\alpha) \\ & - \sinh (\alpha z) \sinh (\alpha) \sin (\alpha) \cos (\alpha z)) \\ & - 2\alpha^2 \beta^2 (\cosh (\alpha z) \cos (\alpha z) \cos (\alpha) \cosh (\alpha) \\ & + \sinh (\alpha z) \sin (\alpha z) \sin (\alpha) \sinh (\alpha)) \\ & - 2\alpha^3 \beta^3 (\cosh (\alpha) \cos (\alpha) \cos (\alpha z) \sinh (\alpha z) \\ & + \sin (\alpha) \sin (\alpha z) \sinh (\alpha z) \cosh (\alpha) \\ & - \cos (\alpha) \cosh (\alpha) \cosh (\alpha z) \sin (\alpha z) \\ & + \sin (\alpha) \sinh (\alpha) \sinh (\alpha z) \cos (\alpha z)) \\ & + (\sinh (\alpha) \sin (\alpha) \cosh (\alpha z) \cos (\alpha z) \\ & - \cos (\alpha) \cosh (\alpha) \sin (\alpha z) \sinh (\alpha z))] / [\alpha^2 (1 + 4\alpha^4 \beta^4) \\ & (\sinh^2 \alpha + \cos^2 \alpha)] \quad (10) \end{aligned}$$

$$\begin{aligned} \frac{u_v}{P_0} = & \left\{ [\alpha \beta (\cosh (\alpha) \cos (\alpha) \cos (\alpha z) \sinh (\alpha z) \right. \\ & + \sin (\alpha z) \sinh (\alpha z) \sin (\alpha) \cosh (\alpha) \\ & - \cos (\alpha) \cosh (\alpha) \sin (\alpha z) \cosh (\alpha z) \end{aligned}$$

$$\begin{aligned}
& + \sin(\alpha) \sinh(\alpha) \sinh(\alpha z) \cos(\alpha z)) \\
& - 2\alpha^2 \beta^2 (\sinh(\alpha z) \sin(\alpha z) \cos(\alpha) \cosh(\alpha) \\
& - \sin(\alpha) \sinh(\alpha) \cos(\alpha z) \cosh(\alpha z)) \\
& - 2\alpha^3 \beta^3 (\cos(\alpha) \cosh(\alpha) \cos(\alpha z) \sinh(\alpha z) \\
& + \sin(\alpha) \sin(\alpha z) \sinh(\alpha z) \cosh(\alpha) \\
& + \sin(\alpha z) \cosh(\alpha z) \cos(\alpha) \cosh(\alpha) \\
& - \sin(\alpha) \sinh(\alpha) \sinh(\alpha z) \cos(\alpha z)) \\
& + (\cos(\alpha) \cosh(\alpha) \cos(\alpha z) \cosh(\alpha z) \\
& + \sin(\alpha) \sinh(\alpha) \sin(\alpha z) \sinh(\alpha z))] \\
& / [\alpha^2 (1 + 4\alpha^4 \beta^4) (\sinh^2 \alpha + \cos^2 \alpha)] \Bigg\} - \frac{1}{\alpha^2}
\end{aligned} \tag{11}$$

where

$$P_0 = \frac{1}{2} \alpha^2 P \tag{12}$$

Now, for small α , we expand in powers of α ,

$$\begin{aligned}
\frac{u_x}{P_0} = & (1 - 2\beta^2 - 2\beta z - z^2) + \alpha^4 [(8\beta^6 - 8\beta^4 - 4\beta^3 \\
& - \frac{2}{3} \beta^2 - \frac{7}{3} \beta - \frac{61}{90}) + (8\beta^5 - 4\beta^3 - \frac{4}{3} \beta) z \\
& - (-4\beta^4 + 2\beta^2 - \frac{5}{6}) z^2 - \frac{2}{3} \beta (1 - 2\beta^2) z^3 \\
& - \frac{(1-2\beta^2)}{6} z^4 + \beta \frac{z^5}{15} + \frac{z^6}{90}] + \alpha^8 [(-32\beta^{10}
\end{aligned}$$

$$\begin{aligned}
& + 32\beta^8 + 16\beta^7 - \frac{16}{3}\beta^6 + \frac{4}{3}\beta^5 + \frac{28}{3}\beta^4 + \frac{10}{3}\beta^3 \\
& - \frac{31405}{180 \times 73}\beta^2 - \frac{\beta}{90} + \frac{50521}{1800 \times 63}) + (- 32\beta^9 + 32\beta^7 \\
& + 16\beta^6 - \frac{16}{3}\beta^5 + \frac{4}{3}\beta^4 + \frac{18}{3}\beta^3 - \frac{17441}{180 \times 63}\beta) z \\
& + (16\beta^6 - 16\beta^8 + 8\beta^5 - \frac{8}{3}\beta^4 + \frac{2}{3}\beta^3 + \frac{9}{3}\beta^2 - \frac{16}{45}\beta \\
& - \frac{1385}{360 \times 7}) z^2 - \frac{1}{3} (16\beta^7 - 16\beta^5 - 8\beta^4 - \frac{4}{3}\beta^3 \\
& - \frac{14}{3}\beta^2 - \frac{7}{3}\beta) z^3 - \frac{1}{6} (8\beta^6 - 8\beta^4 - 4\beta^3 - \frac{2}{3}\beta^2 \\
& - \frac{7}{3}\beta - \frac{61}{90}) z^4 - \frac{1}{30} (8\beta^5 - 4\beta^3 - \frac{4}{3}\beta) z^5 \\
& + \frac{1}{90} (- 4\beta^4 + 2\beta^2 - \frac{5}{6}) z^6 + \frac{\beta (1-2\beta^2)}{45 \times 7} z^7 \\
& + \frac{(1-2\beta^2)}{360 \times 7} z^8 - \frac{\beta z^9}{180 \times 63} - \frac{z^{10}}{1800 \times 63}] + \dots
\end{aligned} \tag{13}$$

$$\frac{u_y}{P_0} = \alpha^2 [(2\beta^2 - 4\beta^4 - \frac{5}{6}) = 2\beta (1-2\beta^2) z$$

$$+ (1-2\beta^2) z^2 - \frac{2\beta}{3} z^3 - \frac{1}{6} z^4] + \alpha^6 [(16\beta^8$$

$$\begin{aligned}
& -16\beta^6 - 8\beta^5 + \frac{8}{3}\beta^4 - \frac{2}{3}\beta^3 - \frac{9}{3}\beta^2 + \frac{16}{45}\beta \\
& + \frac{1385}{360 \times 7}) + (16\beta^7 - 16\beta^5 - 8\beta^4 - \frac{4}{3}\beta^3 - \frac{14}{3}\beta^2 \\
& - \frac{7}{3}\beta) z + (8\beta^6 - 8\beta^4 - 4\beta^3 - \frac{2}{3}\beta^2 - \frac{7}{3}\beta - \frac{61}{90}) z^2 \\
& + \frac{1}{3}(8\beta^5 - 4\beta^3 - \frac{4}{3}\beta) z^3 - \frac{1}{6}(2\beta^2 - 4\beta^4 - \frac{5}{6}) z^4 \\
& - \frac{\beta(1-2\beta^2)}{15} z^5 - \frac{(1-2\beta^2)}{90} z^6 + \frac{\beta z^7}{45 \times 7} + \frac{z^8}{360 \times 7}] + \dots (14)
\end{aligned}$$

In the absence of rotation, the problem reduces to the flow through a straight channel under a constant pressure gradient in slip-flow regime, *i e.*,

$$u_x = P_0 (1 - 2\beta^2 - 2\beta z - z^2), u_y = 0 \quad (15)$$

The flux Q_x per unit breadth in the direction of the x -axis is given, for small α , by

$$\begin{aligned}
Q_x &= \int_{-1}^1 u_x dz \\
&= 2P_0 \left(\frac{2}{3} - \beta - 2\beta^2 \right) + 2P_0 \alpha^4 \left(8\beta^6 + 4\beta^5 \right. \\
&\quad \left. - \frac{20}{3}\beta^4 - \frac{17}{3}\beta^3 - \frac{19}{15}\beta^2 - \frac{142}{45}\beta - \frac{136}{315} \right) \\
&\quad + 2P_0 \alpha^8 \left(-32\beta^{10} - 16\beta^9 + \frac{80}{3}\beta^8 + \frac{92}{3}\beta^7 \right.
\end{aligned}$$

$$\begin{aligned}
& + \frac{116}{15} \beta^6 + \frac{118}{45} \beta^5 + \frac{1054}{105} \beta^4 + \frac{1663}{252} \beta^3 \\
& - \frac{3716}{2835} \beta^2 - \frac{69831}{1800 \times 63} \beta + \frac{44224}{155925}) + \dots \quad (16)
\end{aligned}$$

The drag D_x in the direction of the x -axis per unit area on the boundary $z=1$ is found to be

$$\begin{aligned}
& 2P_0 (1+\beta) + 2P_0 \alpha^4 (-4\beta^5 - 4\beta^4 + \frac{4}{3} \beta^2 + \frac{3}{2} \beta - \frac{8}{15}) \\
& + 2P_0 \alpha^8 (16\beta^9 + 16\beta^8 - 8\beta^7 - \frac{64}{3} \beta^6 - \frac{38}{3} \beta^5 \\
& - \frac{68}{15} \beta^4 - \frac{269}{45} \beta^3 - \frac{354}{63} \beta^2 - \frac{10879}{11340} \beta + \frac{992}{2835}) + \dots \quad (17)
\end{aligned}$$

Similarly, the drag D_x at $z = -1$ is

$$\begin{aligned}
& 2P_0 (1-\beta) + 2P_0 \alpha^4 (4\beta^5 - 4\beta^4 + \frac{4}{3} \beta^2 - \frac{3}{2} \beta - \frac{8}{15}) \\
& + 2P_0 \alpha^8 (-16\beta^9 + 16\beta^8 + 8\beta^7 - \frac{16}{3} \beta^6 - \frac{10}{3} \beta^5 \\
& + \frac{72}{16} \beta^4 + \frac{89}{45} \beta^3 - \frac{10}{9} \beta^2 + \frac{1303}{11340} \beta + \frac{992}{2835}) + \dots \quad (18)
\end{aligned}$$

The resistance coefficient becomes approximately

$$\begin{aligned}
\frac{\gamma_r}{\gamma} &= \left(\frac{Q}{Q_x} \right)^2 \\
&= 1 - 2\alpha^4 A + \alpha^8 (2A^2 - 2B) + \dots \quad (19)
\end{aligned}$$

where

$$\begin{aligned}
 A = & \left(8\beta^6 + 4\beta^5 - \frac{20}{3}\beta^4 - \frac{17}{3}\beta^3 - \frac{19}{15}\beta^2 \right. \\
 & \left. - \frac{142}{45}\beta - \frac{136}{315} \right) / \left(\frac{2}{3} - \beta - 2\beta^2 \right), \\
 B = & \left(-32\beta^{19} - 16\beta^9 + \frac{80}{3}\beta^8 + \frac{92}{3}\beta^7 + \frac{116}{15}\beta^6 \right. \\
 & + \frac{118}{45}\beta^5 + \frac{1054}{105}\beta^4 + \frac{1663}{252}\beta^3 - \frac{3716}{2835}\beta^2 \\
 & \left. - \frac{69831}{18006 \times 3}\beta + \frac{44124}{155925} \right) / \left(\frac{2}{3} - \beta - 2\beta^2 \right) \quad (20)
 \end{aligned}$$

γ and γ_r being the resistance coefficients for the flows without and with rotation, respectively, and Q and Q_x being the flux in these cases.

When Ω_0 is large, from (10) and (11), we obtain, for

$$1 \geq z > 0,$$

$$\begin{aligned}
 u_x \simeq & \frac{P_0 e^{\alpha(z-1)}}{\alpha^2 (1 + 4\alpha^4 \beta^4)} \left[(1 + \alpha\beta - 2\alpha^3 \beta^3) \sin \alpha (1-z) \right. \\
 & \left. - (\alpha\beta - 2\alpha^2 \beta^2 - 2\alpha^3 \beta^3) \cos \alpha (1-z) \right] \quad (21)
 \end{aligned}$$

$$\begin{aligned}
 u_y \simeq & \frac{P_0}{\alpha^2} \left[-1 + \frac{e^{\alpha(z-1)}}{(1 + 4\alpha^4 \beta^4)} \left\{ (\alpha\beta + 2\alpha^2 \beta^2) \right. \right. \\
 & + 2\alpha^3 \beta^3 \sin \alpha (1-z) + (1 + \alpha\beta \\
 & \left. \left. - 2\alpha^3 \beta^3) \cos \alpha (1-z) \right\} \right] \quad (22)
 \end{aligned}$$

Similar expressions may be written for $0 > z \geq -1$.

3. DISCUSSION AND CONCLUSIONS

The graphs of u_x and u_y are plotted in Figures 1 and 2. Figure 1 shows that, as β increases, the primary flow u_x decreases. From Figure 2, it is inferred that the secondary flow u_y also decreases in magnitude with an increase in β .

When there is no rotation, the flux in the x -direction and the resistance coefficient are found to be independent of slip parameter. However, in the presence of rotation, as β increases (i) the flux decreases, and (ii) the resistance coefficient increases. For fixed β , as α increases (i) the flux Q_x decreases, and (ii) the resistance coefficient increases. For fixed rotation, as β increases the drag at the upper wall increases, while at the lower wall it decreases. As α increases, the drag at both the walls decreases. It should be pointed out that these results hold true for small values of rotation only.

4. SUGGESTIONS FOR EXPERIMENTAL VERIFICATIONS

It may be possible to perform experiments by rotating a channel of finite width B which is great compared with the depth $2L$. In such a channel, the conditions close to the walls $z' = \pm L$ are not given by the above calculations, but if the side walls are in such a direction that there is no total flow across them, then the conditions can be attained approximately over most of the channel. It is necessary to keep the side walls at an angle $-\Psi$ with the x -axis, where

$$\tan \Psi = \frac{- \int_{-1}^1 u_y dz}{\int_{-1}^1 u_x dz}$$

$$\begin{aligned}
&= [- (\beta - 2\alpha^2\beta^3) (\sin (\alpha) \cosh (\alpha) - \sinh (\alpha) \cos (\alpha)) \\
&\sin (\alpha) \cosh (\alpha) + 2\alpha\beta^2 \sin (\alpha) \cos (\alpha) \\
&- \sinh (\alpha) \cosh (\alpha)) - 1/\alpha (\sin (\alpha) \cos (\alpha) \\
&+ \sinh (\alpha) \cosh (\alpha)) + 2 (1 + 4\alpha^4\beta^4) \\
&(\sinh^2 \alpha + \cos^2 \alpha))] / \\
&[- (\beta + 2\alpha^2\beta^3 (\sin (\alpha) \cosh (\alpha) - \sinh (\alpha) \cos (\alpha) \\
&\sin (\alpha) \cosh (\alpha)) - 2\alpha\beta^2 (\sin (\alpha) \cos (\alpha) \\
&+ \sinh (\alpha) \cosh (\alpha)) + 1/\alpha (\sinh (\alpha) \cosh (\alpha) \\
&- \sin (\alpha) \cos (\alpha)))] . \tag{23}
\end{aligned}$$

Figure 3 gives the angle β versus α for $\beta = 0$ and 0.05 . From this graph, we conclude that the angle Ψ for any α increases as β increases. Thus the effect of the slip parameter does not inhibit the secondary flow through the side walls.

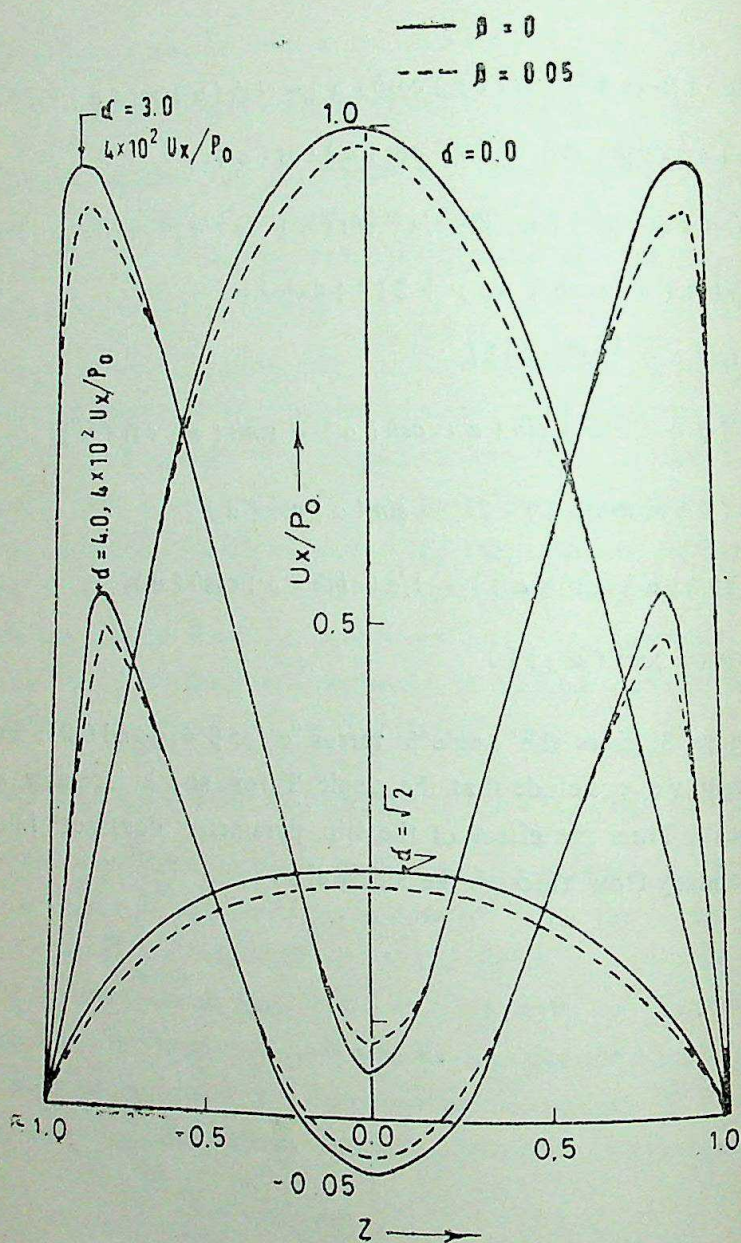


FIG 1 VELOCITY IN THE DIRECTION OF PRESSURE GRADIENT

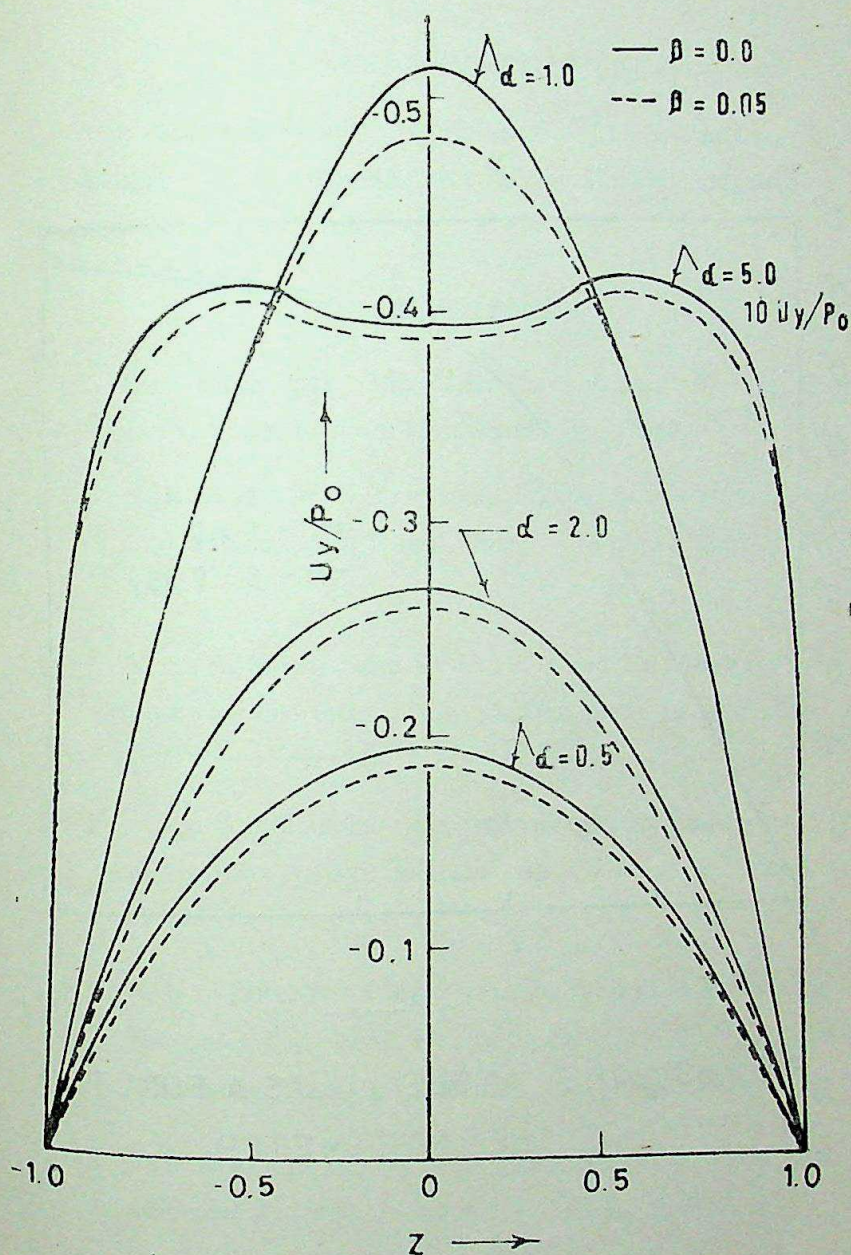


FIG. 2 VELOCITY OF SECONDARY FLOW

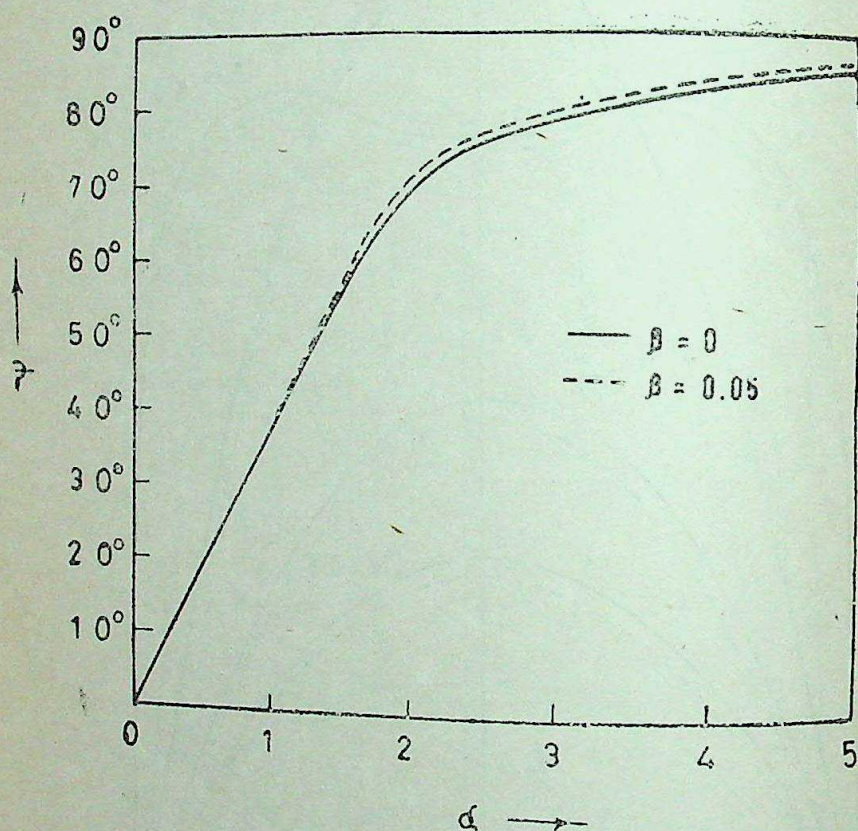


FIG. 3 THE ANGLE ψ AS A FUNCTION OF THE PARAMETER α

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ON A CLASS OF ANALYTIC FUNCTIONS DEFINED BY USING HADAMARD PRODUCT*

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ABSTRACT

We introduce a class, namely $R_{\lambda, \alpha}^{\delta} (A, B)$ of functions defined by using Hadamard product $f * S_{\sigma}(z)$ of $f(z)$ and $S_{\alpha}(z) = \frac{z}{(1-z)^{2(1-\alpha)}}$.

The aim of the present paper is to determine sharp coefficient estimates and maximization theorem concerning the coefficients. Further, we give a sufficient condition in terms of coefficients for functions belonging to the class $R_{\lambda, \alpha}^{\delta} (A, B)$.

1. INTRODUCTION

Let A be the class of functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ which are analytic in the unit disc $U = \{z : |z| < 1\}$. If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ belong to A , the convolution or Hadamard product of $f(z)$ and $g(z)$ is defined by the power series

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$$f * g(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n \quad (1.1)$$

Let S denote the subclass of A consisting of analytic and univalent functions $f(z)$ in the unit disc U . A function $f(z)$ of S is said to be starlike of order α , $0 \leq \alpha < 1$, if

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > \alpha, \quad z \in U. \quad (1.2)$$

We denote by $S^*(\alpha)$ the class of all starlike functions of order α .

Now, the function

$$S_\alpha(z) = \frac{z}{(1-z)^{2(1-\alpha)}} \quad (1.3)$$

is the well known extremal function for the class $S^*(\alpha)$. $S_\alpha(z)$ can be written in the form

$$S_\alpha(z) = z + \sum_{n=2}^{\infty} c(\alpha, n) z^n \quad (1.4)$$

$$\text{where } c(\alpha, n) = \frac{\prod_{k=2}^n (k-2\alpha)}{(n-1)!}, \quad n = 2, 3, \dots \quad (1.5)$$

Then we note that $c(\alpha, n)$ is decreasing in α and satisfies

$$\begin{aligned} & \infty, \text{ when } \alpha < \frac{1}{2} \\ \lim_{n \rightarrow \infty} c(\alpha, n) &= 1, \text{ when } \alpha = \frac{1}{2} \\ & 0, \text{ when } \alpha > \frac{1}{2} \end{aligned} \quad (1.6)$$

In the present note, we introduce a class $R_{\lambda, \alpha}^\delta(A, B)$ of functions $f(z)$ belonging to A and satisfying the condition

$$\left| \frac{e^{i\delta} (f * S_{\alpha}(\alpha))' - 1}{\lambda(A-B) \cos \delta - B e^{i\delta} (f * S_{\alpha}(z))' - 1} \right| < 1, z \in U, \quad (1.7)$$

where A, B, λ, δ are arbitrary numbers such that $-1 \leq B < A \leq 1$, $0 < \lambda \leq 1$, $\delta \in (-\pi/2, \pi/2)$. It is easy to see that the condition (1.7) is equivalent to

$$e^{i\delta} \cdot \{f * S_{\alpha}(z)\}' = (1-\lambda) e^{i\delta} + \cos \delta \cdot \left[\frac{1+A\omega(z)}{1+B\omega(z)} \right] + i\lambda \sin \delta, \quad (1.8)$$

where $\omega(z)$ is analytic in U and satisfies $\omega(0) = 0$, $|\omega(z)| < 1$.

The main aim of the present paper is to study coefficient estimates, maximization of $|a_3 - \mu a_2^2|$ over the class $R_{\lambda, \alpha}^{\delta}(A, B)$ and a sufficient condition in terms of coefficients for a function belonging to $R_{\lambda, \alpha}^{\delta}(A, B)$. All the results are sharp. Our results generalize the corresponding results of Caplinger and Causey [1], Juneja and Mogra [2], Padmanabhan [5], and Shukla and Dashrath [6].

2. LEMMA

The following lemma is due to Keogh and Merkes [3]

Lemma 2.1. Let $\omega(z) = \sum_{k=1}^{\infty} c_k z^k$ be analytic in U such that $\omega(0) = 0$

and $|\omega(z)| < 1$. If d is any complex number, then

$$|c_2 - d c_1^2| \leq \max[1, |d|]. \quad (2.1)$$

Equality may be attained with the functions $\omega(z) = z^2$ and $w(z) = z^2$.

3. COEFFICIENT ESTIMATES

Theorem 3.1. If $f(z)$ is in $R_{\lambda, \alpha}^{\delta}(A, B)$, then

$$|a_j| \leq \frac{(A-B) \cos \delta}{j \cdot c(\alpha, j)}, \quad j=2, 3, 4, \dots, \quad (3.1)$$

$$\text{where } c(\alpha, j) = \frac{\prod_{k=2}^j (k-2\alpha)}{(j-1)!}.$$

The estimate (3.1) is sharp.

Proof. Since $f(z) \in R_{\lambda, \alpha}^{\delta}(A, B)$, we have from (1.8) that

$$e^{i\delta} \cdot (f * S_{\alpha}(z))' = (1-\lambda) e^{i\delta} + \cos \delta \cdot \left[\frac{1+A\omega(z)}{1+B\omega(z)} \right] + i\lambda \sin \delta \quad (3.2)$$

where $\omega(z)$ is analytic in U and satisfies the conditions $\omega(0) = 0$, $|\omega(z)| < 1$ for $z \in U$. From (3.2), we have

$$e^{i\delta} \cdot (f * S_{\alpha}(z))' - 1 = [(A-B)\lambda \cos \delta - B e^{i\delta} (f * S_{\alpha}(z))' - 1] \omega(z)$$

or

$$e^{i\delta} \cdot \sum_{n=2}^{\infty} n \cdot c(\alpha, n) a_n z^{n-1} = [(A-B)\lambda \cos \delta - B e^{i\delta} \cdot \sum_{n=2}^{\infty} n \cdot c(\alpha, n) a_n z^{n-1}] \omega(z), \quad (3.3)$$

$$\text{where } \omega(z) = \sum_{n=1}^{\infty} c_n z^n.$$

Equating corresponding coefficients on both sides of (3.3), we find that the coefficient a_j on the left hand side of (3.3) depends on $a_2, a_3, \dots, a_{j-1}$ on the right hand side of (3.3). Hence, for $j \geq 2$, it follows from (3.3) that

$$\begin{aligned}
& c^{i\delta} \cdot \sum_{n=2}^j n \cdot c(\alpha, n) a_n z^{n-1} + \sum_{n=j+1}^{\infty} a_n z^{n-1} \\
& = [\lambda (A-B) \cos \delta - B e^{i\delta} \sum_{n=2}^{j-1} n \cdot c(\alpha, n) a_n z^{n-1}] \omega(z)
\end{aligned}$$

where a_n are some complex numbers.

Since $|\omega(z)| < 1$, by using parseval's identity,

$$\begin{aligned}
& \sum_{n=2}^j n^2 \cdot (c(\alpha, n))^2 \cdot |a_n|^2 \cdot r^{2(n-1)} + \sum_{n=j+1}^{\infty} |a_n|^2 \cdot r^{2(n-1)} \\
& \leq \lambda^2 (A-B)^2 \cos^2 \delta + B^2 \cdot \sum_{n=2}^{j-1} n^2 \cdot (c(\alpha, n))^2 \cdot |a_n|^2 \cdot r^{2(n-1)} \\
& \leq \lambda^2 (A-B)^2 \cos^2 \delta + B^2 \cdot \sum_{n=2}^{j-1} n^2 (c(\alpha, n))^2 \cdot |a_n|^2 \cdot |a_n|^2.
\end{aligned}$$

Letting $r \rightarrow 1$ on the left hand side of the above inequality, we obtain

$$\begin{aligned}
& \sum_{n=2}^j n^2 \cdot (c(\alpha, n))^2 \cdot |a_n|^2 \\
& \leq \lambda^2 (A-B)^2 \cos^2 \delta + B^2 \cdot \sum_{n=2}^{j-1} n^2 \cdot (c(\alpha, n))^2 |a_n|^2.
\end{aligned}$$

Thus

$$\begin{aligned}
& j^2 \cdot (c(\alpha, j))^2 \cdot |a_j|^2 \leq \lambda^2 (A-B)^2 \cos^2 \delta - \\
& - (1-B^2) \sum_{n=2}^{j-1} n^2 (c(\alpha, n))^2 |a_n|^2 \leq \lambda^2 (A-B)^2 \cos^2 \delta.
\end{aligned}$$

Hence

$$|a_j| \leq \frac{(A-B) \cos \delta}{j \cdot c(\alpha, j)}, \quad j=2,3, \dots,$$

$$\text{where } c(\alpha, j) = \frac{\prod_{k=2}^j (k-2\alpha)}{(j-1)!}.$$

To show the sharpness of the result, we consider the function $f(z)$ defined by

$$(f * S_\alpha(z))' = 1 + \frac{\lambda(A-B) \cos \delta \cdot e^{-i\delta} \cdot z^{n-1}}{1 + Bz^{n-1}}. \quad (3.1)$$

It is now easy to see that the equality is attained in (3.1).

THEOREM 3.2. If $f(z) \in R_{\lambda, \alpha}^\delta(A, B)$.

Then

$$(1-B)^2 \sum_{n=2}^{\infty} n^2 \cdot (c(\alpha, n))^2 \cdot |a_n|^2 \leq \lambda^2 (A-B)^2 \cos^2 \delta \quad (3.5)$$

Proof. Since $f(z) \in R_{\lambda, \alpha}^\delta(A, B)$, Then we have from (1.8) that

$$e^{i\delta} \cdot (f * S_\alpha(z))' = (1-\lambda) e^{i\delta} + \cos \delta \cdot \left[\frac{1 + A\omega(z)}{1 + B\omega(z)} \right] + i\lambda \sin \delta$$

where $\omega(z) = \sum_{n=1}^{\infty} c_n z^n$ is analytic in U and satisfies

$\omega(0) = 0$ and $|\omega(z)| < 1$ for $z \in U$. Hence

$$e^{i\delta} \cdot \sum_{n=2}^{\infty} n \cdot c(\alpha, n) a_n z^{n-1} = [\lambda(A-B) \cos \delta - B e^{i\delta} \cdot \sum_{n=2}^{\infty} n \cdot c(\alpha, n) a_n z^{n-1}] \omega(z),$$

On solving, we get

$$\sum_{n=2}^{\infty} n^2 \cdot (c(\alpha, n))^2 \cdot |a_n|^2$$

$$\leq \lambda^2 (A-B)^2 \cos^2 \delta + B^2 \sum_{n=2}^{\infty} n^2 (c(\alpha, n))^2 |a_n|^2$$

$$(1-B^2) \sum_{n=2}^{\infty} n^2 \cdot (c(\alpha, n))^2 \cdot |a_n|^2 \leq \lambda^2 (A-B)^2 \cos^2 \delta.$$

THEOREM 3.3. Let $f(z)$ be analytic in U . If for $-1 \leq B \leq 0$,

$$\sum_{n=2}^{\infty} (1-B) \cdot (c(\alpha, n) \cdot n |a_n|) \leq \lambda (A-B) \cos \delta, \quad (3.6)$$

then $f(z) \in R_{\lambda, \alpha}^{\delta}(A, B)$.

The result is sharp.

Proof. Suppose that (3.6) holds. Then, for $z \in U$, we have

$$\begin{aligned} & |e^{i\delta} (f + S_{\alpha}(z))' - 1| - |\lambda (A-B) \cos \delta - B e^{i\delta} (f(z)) - 1| \\ &= |e^{i\delta} \sum_{n=2}^{\infty} c(\alpha, n) n a_n z^{n-1}| - |\lambda (A-B) \cos \delta \\ &\quad - B e^{i\delta} \cdot \sum_{n=2}^{\infty} c(\alpha, n) \cdot n a_n z^{n-1}| \\ &\leq \sum_{n=2}^{\infty} c(\alpha, n) n |a_n| r^{n-1} - \lambda (A-B) \cos \delta - B \sum_{n=2}^{\infty} c(\alpha, n) \cdot n |a_n| r^{n-1} \\ &< \sum_{n=2}^{\infty} c(\alpha, n) \cdot n |a_n| - \lambda (A-B) \cos \delta - B \sum_{n=2}^{\infty} c(\alpha, n) \cdot n |a_n| \end{aligned}$$

$$= \sum_{n=2}^{\infty} (1-B) c(\alpha, n) \cdot n |a_n| - \lambda (A-B) \cos \delta \leq 0, \text{ by (3.6).}$$

Hence it follows that

$$\left| \frac{e^{i\delta} \cdot (f^* S_{\alpha}(z))' - 1}{(A-B) \cos \delta - B e^{i\delta} \cdot (f^* S_{\alpha}(z))' - 1} \right| < 1, z \in U.$$

Therefore $f(z) \in R_{\lambda, \alpha}^{\delta}(A, B)$.

The function

$$f(z) = z - \frac{\lambda (A-B) \cos \delta \cdot e^{-i\delta}}{(1-B) \cdot c(\alpha, n) \cdot n} z^n, n=2, 3, \dots$$

is an extremal function for this theorem since for this function

$$\left| \frac{e^{i\delta} (f^* S_{\alpha}(\alpha))' - 1}{\lambda (A-B) \cos \delta - B e^{i\delta} (f^* S_{\alpha}(z))' - 1} \right| < 1,$$

4. MAXIMIZATION OF $|a_3 - \mu a_2^2|$

Theorem 4.1. If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in R_{\lambda, \alpha}^{\delta}(A, B)$, then, for any complex μ , we have

$$|a_3 - \mu a_2^2| \leq \frac{\lambda (A-B) \cos \delta}{3c(\alpha, 3)} \max \{1, |d|\}, \quad (4.1)$$

where

$$d = \frac{4B \sec \delta \cdot e^{i\delta} \cdot (c(\alpha, 2))^2 + 3\mu \lambda (A-B) \cdot c(\alpha, 3)}{4 \sec \delta \cdot e^{i\delta} \cdot (c(\alpha, 2))^2}.$$

The result is sharp .

Proof . Since $f(z) \in R_{\lambda, \alpha}^{\delta}(A, B)$, we have

$$e^{i\delta} \cdot (f * S_{\alpha}(z))' = (1-\lambda) e^{i\delta} + \lambda \cos \delta \cdot \left[\frac{1+A\omega(z)}{1+B\omega(z)} \right] + i \lambda \sin \delta \quad (4.2)$$

where $\omega(z) = \sum_{n=1}^{\infty} c_n z^n$ and satisfies the conditions $\omega(0) = 0$ and

$|\omega(z)| < 1$ for $z \in U$. From (4.2), we get

$$\begin{aligned} \omega(z) &= \frac{e^{i\delta} \cdot (f * S_{\alpha}(z))' - 1}{\lambda(A-B) \cos \delta - B e^{i\delta} (f * S(z))' - 1} \\ &= \frac{\sec \delta \cdot e^{i\delta}}{\lambda(A-B)} \left[2c(a,2) a_2 z + 3c(a,3) a_3 z^2 + \frac{B e^{i\delta} 4(c(a,2))^2 a_2^2 z^2}{\lambda(A-B) \cos \delta} \right]. \end{aligned}$$

Now comparing the coefficients of z and z^2 on both sides, we have

$$c_1 = \frac{\sec \delta \cdot e^{i\delta}}{\lambda(A-B)} 2c(a,2) a_2$$

$$\text{and } c_2 = \frac{\sec \delta \cdot e^{i\delta}}{\lambda(A-B)} \left[3c(a,3) a_3 + \frac{4B e^{i\delta} (c(a,2))^2 a_2^2}{\lambda(A-B) \cos \delta} \right].$$

$$\text{Thus } a_2 = \frac{c_1 \lambda(A-B)}{2 \sec \delta \cdot e^{i\delta} \cdot c(a,2)}$$

$$\text{and } a_3 = \frac{c_2 \lambda(B-B) - B c_1^2}{3 \sec \delta \cdot e^{i\delta} \cdot c(a,3)}.$$

Hence

$$a_3 - \mu a_2^2$$

$$= \frac{c_2 \lambda (A-B) - B c_1^2 \lambda (A-B)}{3 \sec \delta \cdot e^{i\delta} \cdot c(\alpha, 3)} - \mu \cdot \frac{c_1^2 \lambda^2 (A-B)^2}{4 \sec^2 \delta \cdot e^{i\delta} \cdot e^{2i\delta} (c(\alpha, 2))^2}$$

$$= \frac{\lambda (A-B) \cos \delta}{3 e^{i\delta} c(\alpha, 3)} \left[c_2 - c_1^2 \frac{4 B \sec \delta \cdot e^{i\delta} \cdot (c(\alpha, 2))^2 + 3 \mu \lambda (A-B) c(\alpha, 3)}{4 \sec \delta \cdot e^{i\delta} \cdot (c(\alpha, 2))^2} \right].$$

Therefore

$$|a_3 - \mu a_2^2| = \frac{\lambda (A-B) \cos \delta}{3 \cdot c(\alpha, 3)} |c_2 - d c_1^2|,$$

where

$$d = \frac{4 B \sec \delta \cdot e^{i\delta} \cdot (c(\alpha, 2))^2 + 3 \mu \lambda (A-B) \cdot c(\alpha, 3)}{4 \sec \delta \cdot e^{i\delta} \cdot (c(\alpha, 2))^2}.$$

Applying Lemma 2.1. In the above equation, we get,

$$|a_3 - \mu a_2^2| \leq \frac{\lambda (A-B) \cos \delta}{3 c(\alpha, 3)} \max \{1, |d|\}.$$

Since the inequality (2.1) is sharp, so that the result (4.1) is sharp.

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A NOTE ON CERTAIN GENERATING RELATIONS INVOLVING A CLASS OF POLYNOMIALS

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ABSTRACT

The authors aim at deriving generating relations of Mehler's type by applying a group-theoretic approach.

1. INTRODUCTION

In an attempt to unify classical polynomials, Patil and Thakare [1] considered a class of functions defined by

$$(1.1) \quad P_n^\alpha(x; r, s, p, k, \lambda) \\ = x^{-\alpha} e^{px^r} \theta_{x,\lambda}^n [x^{\alpha+sn} e^{-px^r}],$$

where

$$\theta_{x,\lambda} = \lambda x^k + x^{k+1} D, \quad D \equiv \frac{d}{dx}.$$

The work of Patil and Thakare and several others, using the differential operator $\theta_{x,\lambda}$ was motivated by the work of Srivastava and Singhal [3], using the special case $\theta_{x,0} = x^{k+1} D_x$. The above generalization includes classical and other polynomials and functions of mathematical physics as its special cases, like Hermite, Gegenbauer and Laguerre polynomials and their generalizations. One can refer to

Srivastava and Manocha [2, pp. 447-448, Problems 16 and 18] for more general situations in which the operator $\theta_{x,\lambda}$ has already been used.

In the present note, we shall derive Mehler's type formulas for the function (1.1) by group-theoretic approach when a bilinear generating relation is known.

2. LIE OPERATORS FOR $P_n^\alpha (x; r, s, p, k, \lambda)$

Patil and Thakare [1] have given the operational relation

$$(2.1) \quad [\theta_{x,\lambda} + \alpha x^k - prx^{k+r}] P_n^\alpha (x; r, s, p, k, \lambda)$$

$$= x^{-s} P_{n+1}^{\alpha-s} (x; r, s, p, k, \lambda).$$

Now put

$$(2.2) \quad \Lambda_{y,x} = yx^s [\theta_{x,\lambda} + \alpha x^k - prx^{k+r}],$$

so that

$$(2.3) \quad \Lambda_{y,x} \{ y^n P_n^\alpha (x; r, s, p, k, \lambda) \}$$

$$= y^{n+1} P_{n+1}^{\alpha-s} (x; r, s, p, k, \lambda).$$

Hence, clearly $\Lambda_{y,x}$ forms a raising Lie operator, for the class of functions $P_n^\alpha (x; r, s, p, k, \lambda)$.

The extended form of this operator is given by

$$(2.4) \quad e^{\phi \Lambda_{y,x}} f(x; y)$$

$$= (1-t)^{-(a+\lambda)/k} \exp \{ p x^r \{ 1 - (1-t)^{-r/k} \} \} f \{ x (1-t)^{-1/k}; y \},$$

where $t = \phi y k x^{k+s}$.

3. MEHLER'S TYPE FORMULAE

Consider the bilinear generating function

$$(3.1) \quad G(x, z, \phi)$$

$$= \sum_{n=0}^{\infty} a_n \phi^n P_n^{\alpha}(x; r, s, p, k, \lambda) P_n^{\beta}(z; e, f, i, j, \delta).$$

Putting $\phi = \phi t_1 t_2 g$, we get

$$(3.2) \quad G(x, z, \phi t_1 t_2 g)$$

$$= \sum_{n=0}^{\infty} a_n \phi^n g^n \{ t_1^n P_n^{\alpha}(x; r, s, p, k, \lambda) \}$$

$$t_2^n P_n^{\beta}(z; e, f, i, j, \delta) \}.$$

Now operating upon both sides by $\{ (\exp \phi \Lambda_{x, t_1}) (\exp \phi \Lambda_{z, t_2}) \}$ and with appropriate adjustments of parameters, we find, using (2.3), that

$$(3.3) \quad (\exp \phi \Lambda_{x, t_1}) (\exp \phi \Lambda_{z, t_2}) G(x, z, \phi t_1 t_2 g)$$

$$= \sum_{n=0}^{\infty} a_n \sum_{l_1=0}^{\infty} \sum_{l_2=0}^{\infty} \frac{\phi^{l_1}}{l_1!} t_1^{l_1+n} P_{n+l_1}^{\alpha-l_1 s}(x; r, s, p, k, \lambda)$$

$$\frac{\phi^{l_2}}{l_2!} t_2^{l_2+n} P_{n+l_2}^{\alpha-l_2 f}(z; e, f, i, j, \delta) (\phi g).$$

But, by (2.4) ,

$$\begin{aligned}
 (3.4) \quad & (\exp \phi \Lambda_{x, t_1}) (\exp \phi \Lambda_{z, t_2}) G (x, z, \phi t_1 t_2 g) \\
 &= (1 - \phi t_1 k x^{k+s})^{- (\alpha + \lambda) / k} \exp [p x^r \{ 1 - \phi t_1 k x^{s+k} \}^{-r/k}] \\
 & \quad (1 - \phi t_2 j z^{f+j})^{- (\beta + \delta) / j} \exp [i z^e \{ 1 - \phi t_2 j z^{f+j} \}^{-e/j}] . \\
 & G [x (1 - \phi t_1 k x^{k+s})^{-1/k} , z (1 - \phi t_2 j z^{f+j})^{-1/j} , \phi t_1 t_2 g] .
 \end{aligned}$$

From (3.3), and (3.4) on putting $t_1 = t_2 = 1$ and after some manipulations, we get

$$\begin{aligned}
 (3.5) \quad & (1 - \phi k x^{k+s})^{- (\alpha + \lambda) / k} \exp [p x^r \{ 1 - (1 - \phi k x^{s+k}) \}^{-r/k}] \\
 & \quad (1 - \phi j z^{f+j})^{- (\beta + \delta) / j} \exp [i z^e \{ 1 - (1 - \phi j z^{f+j}) \}^{-e/j}] . \\
 & G [x (1 - \phi k x^{k+s})^{-1/k} , z (1 - \phi j z^{f+j})^{-1/j} , \phi g] \\
 &= \sum_{n=0}^{\infty} \sum_{l_1=0}^{\infty} \sum_{l_2=0}^{\min(n, l_1)} \frac{a_{n-l_2} \phi^{n+l_1-l_2} g^{n-l_2}}{(l_1-l_2)! l_2!} .
 \end{aligned}$$

$$P_{n+l_1-2l_2}^{a-(l_1-l_2)s} (x; r, s, p, k, \lambda) P_n^{a-l_2f} (z; e, f, i, j, \delta)$$

Thus we have proved the theorem :

If there exists a bilinear generating function of the form (3.1), then there exists a generating relation of the form (3.5) .

4. APPLICATION OF THE THEOREM

Although the above theorem can be applied to many classical polynomials, we give below a special case involving Hermite polynomials.

We start from the following bilinear generating function for Hermite polynomials $H_n(x)$, known as Mehler's formula :

$$(4.1) \quad \sum_{n=0}^{\infty} H_n(x) H_n(y) \frac{t^n}{2^n n!} \\ = (1-t^2)^{-1/2} \exp \left\{ \frac{2xyt - t^2(x^2 + y^2)}{1-t^2} \right\}.$$

Hence, from (1.1) (3.1) and (3.5), we get

$$(4.2) \quad (1-t^2g^2)^{-1/2} e^{\{x^2-(x+t)^2\}} e^{\{z^2-(z+t)^2\}} \\ \exp \left\{ \frac{2(x+t)(z+t)tg - \{(x+t)^2 + (z+t)^2\}t^2g^2}{1-t^2g^2} \right\} \\ = \sum_{n=0}^{\infty} \sum_{l_1=0}^{\infty} \sum_{l_2=0}^{\min(n, l_1)} \frac{(-1)^{l_1} t^{n+l_1-l_2} g^{n-l_2}}{2^{n-l_2} (n-l_2)! (l_1-l_2)!}$$

$$H_{n+l_1-2l_2}(x) \cdot H_n(z).$$

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A NOTE ON SCATTERED SPACES

By

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The purpose of this note is to present some results on scattered spaces. The principal theorems concern scattered spaces and inverse limits. We shall use the following nomenclature and notation. Let X be a topological space and let $A \subset X$. Then A' is the set of accumulation points of A . The symbol Φ denotes the empty set. Also, we shall assume that all spaces are non-empty.

Definiton . A topological space X is scattered if and only if for any non-empty subset $A \subset X$, A is not contained in A' . That is, there is a point $x \in A$ which is not a accumulation point of A .

A simple example of a compact, scattered space is the set $X = \{0\} \cup \{1/n : n \geq 1\}$ with the relative topology from the reals.

Remark . It is fairly easy to show that a T_1 -space which is scattered is totally disconnected (see [3], page 33). Further, Example 57 of [3] is a connected scattered space. Finally, this remark and Proposition 8.6, page 46, of [2] gives the following

Proposition I. If X is a compact, scattered, Hausdorff space, then $\dim X \leq 0$.

The next example shows that if we drop the assumption that X is compact, then the dimension of X need not be less than or equal to 0 at every point.

Example 1. Define the following three sets in the plane.

Let $X_1 = \{ (0, 1/n) : n \geq 1 \}$, $X_2 = \{ (1/m, 0) : m \geq 1 \}$ and $X_3 = \{ (1/m, 1/n) : m \geq 1, n \geq 1 \}$. Then set $X = \{ (0, 0) \} \cup X_1 \cup X_2 \cup X_3$. Define basic open sets for X as follows. For $(0, 1/n)$ a basic open set has the form $\{ (0, 1/n) \} \cup \{ (1/m, 1/n) : m \geq k \}$ and for $(1/m, 0)$ the basic open sets have the form $\{ (1/m, 0) \} \cup \{ (1/m, 1/n) : n \geq k \}$. Next the sets $\{ (1/m, 1/n) \}$ are all open. Finally, a basic open set of $(0, 0)$ has the form $\{ (0, 0) \} \cup \{ (1/m, 1/n) : n \geq i, m \geq j \}$. Then X is T_2 but not compact. Moreover, each of the points $(0, 1/n)$ or $(1/m, 0)$ is a limit point of X_3 . However, X is scattered, but $\dim X$ is not less than or equal to 0.

Remark. If $\text{ind } X \geq 1$ at each point of a T_2 -space X , then $\{x\}$ is not open for any $x \in X$. ϵX . Thus, $x \in X'$ for all $x \in X$ and X is not scattered.

In this note a directed set $(D, \leq)$ is a non-empty set D and a partial order on D such that if $a, b \in D$, there is a $c \in D$ with $a \leq c$ and $b \leq c$. Let $(D, \leq)$ be a directed set and let X_d be a topological space for each $d \in D$. Further, suppose that for each pair $a, b \in D$, $a \leq b$, there is a function $f_{ab} : X_b \rightarrow X_a$, with f_{aa} the identity on X_a , for all $a \in D$. We shall also assume that if $a \leq c \leq b$, then $f_{ab} = f_{ac} \circ f_{cb}$. Then the collection of spaces X_d , the directed set D and the set of functions form an inverse limit system. Designate an inverse limit system by IS. The inverse limit of the system is the space X_∞ of all members of the product $X = \prod \{ x_d : d \in D \}$ which satisfy $f_{ab}(x_b) = \pi_a(x)$ for all $a, b \in D$ with $a \leq b$. The space X_∞ has the relative product topology. In general, the space X_∞ is quite different from the spaces X_d . In the sequel we shall use π_d to mean the projection restricted to X_∞ when it is applied to X_∞ . A discussion of inverse

limits can be found in [1]. In particular, we have (Theorem 6. b. 11, page 171 of [1]).

Theorem 2. If each of the spaces X_a is compact and T_2 , then X_∞ is non-empty.

Further, we have the elementary

Lemma 3. If $x, y \in X_\infty$ and if $x_b = y_b$ for some $b \in D$, then $x_a = y_a$ for all $a \in D$ with $a < b$.

Proof. This follows from the fact that

$$x_a = f_{ab}(x_b) = f_{ab}(y_b) = y_a.$$

Further, basic open set in X_∞ have a very simple form. In Lemma 4, the projections are restricted to X_∞ .

Lemma 4. Let $B = \{\pi_d^{-1}(U_d) : d \in D, U_d \subset X_d \text{ an open set}\}$.

Proof. A basic open set in the product topology is the intersection of the inverses of an open set in each of a finite number of coordinate spaces. Thus let $d_1, \dots, d_n$ be a finite subset of D and $U_{d_i}, i = 1, \dots, n$, be open sets. Now let $d \in D$ be such that $d_i \leq d$ for all $i = 1, \dots, n$. Then if

$x \in \bigcap \{\pi_{d_i}^{-1}(U_{d_i}) : i = 1, \dots, n\} = U$, set $U_d = \bigcap \{f_{d_i}^{-1}(U_{d_i}) : i = 1, \dots, n\}$, then $x_d \in U_d$ and $\pi_d^{-1}(U_d) \subset U$. Thus, the collection B is a basis for the topology on X_∞ . (Also, see Theorem [6 B.B, page 171 of [1]).

We shall also need the following

Lemma 5. Let X, Y be topological spaces and let $f: X \rightarrow Y$ be a continuous function. If $x \in A \subset X, f(x) = y$ and if y is not an

accumulation point of $f(A)$, then there is an open set U , $x \in U$, such that $f(A \cap U) = \{y\}$.

Proof. Since y is not an accumulation point of $f(A)$, there is an open set V , $y \in V$, such that $f(A) \cap V = \{y\}$. Let U , $x \in U$, be an open set in X such that $f(U) \subset V$. Then $f(A \cap U) \subset f(A) \cap f(U) \subset f(A) \cap V = \{y\}$.

We shall use the following definition in the first of the principal results.

Definition. An inverse limit system IS *splits* at the point x_a in the space X_a if and only if given $b \in D$, $a < b$, $f_{ab}^{-1}(x_a) \neq \emptyset$ and if $x_b \in f_{ab}^{-1}(x_a)$, there is a point $c \in D$ with $b < c$ such that $f_{cb}^{-1}(x_b)$ contains at least two points.

We shall need the next lemma for the first main theorem.

Lemma 6. Suppose that each of the spaces X_d is compact and T_2 and suppose that the system IS splits at the point x_a in X_a . Then the set $A(x_b) = \{x \in X_\infty : \pi_b(x) = x_b\} \neq \emptyset$ where $d \in D$ with $a \leq d$ and $x_b \in f_{ab}^{-1}(x_a)$.

Proof. Let X denote the cartesian product of the spaces X_d , $d \in D$. Let b be such that $a < b$ and let $x_b \in f_{ab}^{-1}(x_a)$ be fixed. Then for $c, b \leq c$, set $A_c(x_b) = \{x \in X : \pi_b(x) = x_b \text{ and } \pi_d(x) = f_{dc}(\pi_b(x)) \text{ for all } d < c\}$. Note that if $x \in f_{be}^{-1}(x_b)$, we can define a point x by

setting $\pi_c(x) = x_c$ and $\pi_d(x) = f_{dc}(x_c)$ for $d < c$. If $d \in D$ is not less than or equal to c , the set $\pi_d(x)$ equal to any point in X_d . Thus, the sets $A_c(x_b)$ are non-empty. If x is not an element of $A_c(x_b)$, then there is a $d \leq c$ such that $\pi_d(x) \neq f_{dc}(\pi_c(x))$. Then there are two disjoint open sets U, V in X_d such that $\pi_d(x) \in U$ and $f_{dc}(\pi_c(x)) \in V$. Let $W \subset X$ be $\pi_c^{-1}(f_{dc}^{-1}(V)) \cap \pi_d^{-1}(U)$. We see that $x \in W$ and that $W \cap A_c(x_b) = \Phi$. Hence, the sets $A_c(x_b)$ are closed. Moreover, if $b < d < c$, then $A_c(x_b) \subset A_d(x_g)$ and thus, the collection $\{A_c(x_b) : b < c\}$ has the finite intersection property. Therefore the set $A(x_b) = \bigcap \{A_c(x_b) : b < c\} \neq \Phi$.

Our first result concerning inverse limits and scattered spaces is the following.

Theorem 7. Suppose that each of the spaces $X_d, d \in D$ is compact. T_2 . If the inverse system IS splits at $x_a \in X_a$ for some $a \in D$, then there is a set $A \subset X_\infty$ such that $A \subset A'$.

Proof. Let $A = A(x_a)$ from Lemma 6. Let $x \in A$. Set $x_d = \pi_d(x)$ for all $d \in D$. Now let U be a basic open set which contains x . Then by Lemma 4, $U = \pi_b^{-1}(U_b)$ where U_b is an open subset of x_b which contains x_b . If $b \leq a$, then $\pi_b(y) = f_{ba}(\pi_a(y)) = f_{ba}(x_a) = x_b$ for all $y \in A$. Hence, $A \subset U$. So, we may assume that $a \leq b$. Now, since the system splits at x_a , there is a $b < c$, such that $f_{bc}^{-1}(x_b)$ contains at least two points. Thus, from the construction in the proof of Lemma 6, we see that there is a point $y \neq x$ in A and $y_c \in f_{bc}^{-1}(x_b)$. Since $y_b = x_b, y \in U$. Thus, $x \in A'$.

Corollary 8. If each X_a is compact, T_2 and if the system splits at some x_a , then X_∞ is not scattered.

If the directed set has a largest element ω , then many of the properties of X_∞ are determined by the properties of X_ω . In particular we have

Theorem 9. If $\omega \in D$ is the largest element of D and if X_ω is scattered, then X_∞ is scattered.

Proof. Let A be a non-empty subset of X_∞ and let $A_\omega = \pi_\omega(A)$. Since X_ω is scattered, there is a $x_\omega \in A_\omega$ which is not an element of A'_ω . Let U_ω be an open set with $x_\omega \in U_\omega$ such that $U_\omega \cap A_\omega = \{x_\omega\}$ and let $x \in A$ with $\pi_\omega(x) = x_\omega$. Then by Lemma 5 there is an open set $V \subset X_\infty$ such that $x \in V \subset U$ and $V \cap A = \{x\}$. Hence, x is not an element of A' , and so, X_∞ is scattered.

Next we need the following definition.

Definition. The inverse system IS is eventually one-to-one if and only if for every $a \in D$ and $x_a \in X_a$, there is a $b \in D$ $a < b$, such that $f_{cb}^{-1}(x_b)$ contains at most one point for all $x_b \in f_{ab}^{-1}(x_a)$ and for all $c \in D$ with $b < c$.

Now we have a partial converse of Theorem 7.

Theorem 10 If the system IS is eventually one-to-one and if each of the spaces X_a , $a \in D$, is scattered, then X_∞ is scattered.

Proof. Let A be a non-empty subset of X_∞ , let $a \in D$ and set $A_a = \pi_a(A)$. Since X_a is scattered, there is an $x_a \in A_a - A'_a$. Thus, there is an open set U_a of X_a such that $U_a \cap A_a = \{x_a\}$. Next there is a $b \in D$, $a < b$, such that for any $x_b \in f_{ab}^{-1}(x_a)$, and any c in D ,

$b < c$, $f_{bc}^{-1}(x_b)$ contains at most one point. Set $B = \pi_b(A) \cap f_{ab}^{-1}(x_a)$. Then $\pi_b(x) \in B$ where $x \in \pi_a^{-1}(x_a) \cap A$. Since X_b is scattered, there is a point $x_b \in B$ and an open set U_b of X_b such that $U_b \cap B = \{x_b\}$. Further, U_b can be chosen so that $f_{ab}(U_b) \subset U_b$. Next set $U = \pi_b^{-1}(U_b)$. If $y \in U \cap A$, then $y_b = \pi_b(y) \in B$. Therefore $y_b = x_b$ since $\pi_a(y) = x_a$. If $z \in U \cap A$, then $z_b = x_b$. Further, if $c \in D$, $b < c$, then $f_{cb}^{-1}(x_b)$ contains at most one point and hence, $z_c = y_c$. Moreover, if $d < b$, $z_d = y_d$ by Lemma 3. Finally, if d is any element of D , there is an element b in D such that $b < k$ and $d < k$. But then $y_d = f_{kd}(y_k) = f_{kd}(z_k) = x_k$. So $z = y$ and $U \cap A = \{y\}$. Therefore y is not in A' and X_∞ is scattered.

Note. A preliminary version of this paper was presented to the ICM-86 in Berkeley, California, August 1986.

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CAPILLARY INSTABILITY OF A STREAMING HOLLOW CYLINDER

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ABSTRACT

The capillary instability of a hollow jet flow of a streaming liquid surrounding a gas cylinder (the equilibrium gas pressure is held constant) subject to its liquid inertia force is presented . A general eigenvalue relation is derived and studied analytically, and the results are confirmed numerically. A comparison is made between the present results and those for the nonhollow-cylinder . The streaming has always a destabilizing influence . The latter attains its maximum value if the basic flow streams in the opposite z -direction . In the absence of streaming, the model is stable for all non-axisymmetric modes for all wavelengths while to axisymmetric mode it is stable (or not) if the perturbed wavelength is equal to or shorter (or longer) than the circumference of the cylinder . The maximum temporal amplification prevailing in the nonhollow case is far lower than of the present model ; this is physically interpreted here .

1. INTRODUCTION

Crucial interest in the linear ([6], [8], [1]) and nonlinear ([10], [4], [5], [2]) hydrodynamic instability of cylindrical jets has been revised due to certain astronomical and industrial problems for many

applications ranging from the design of sprays to the design of ink jet printers .

In recent years studies on the hollow cylinder stability have been flourishing. In view of its practical applications in the astrophysics science ; Kendall [3] has performed interesting experiments with modern equipment to the annular jet (gas cylinder submerged in liquid jet) under the action of surface tension and inertia forces provided that the inertia liquid force is paramount over that of the gas cylinder . Moreover, Kandell [3] called attention to analytically studying the stability of such models . He mentioned : "it appears that the hollow jet instability and encapsulation process being studied here have not been repeated in the literature" (see [3. p. 2086]) .

The purpose of the present work is to investigate the instability of a streaming hollow cylinder endowed with surface tension and acting upon its liquid inertia force . It is worthwhile to mention here that the phenomenon of the hollow cylinder may occur in the crust of the earth when a gas escaping from below oil layers or in the sea during geological drillings or when air is pumped into the water .

2. FORMULATION OF THE PROBLEM AND EIGENVALUE RELATION

Let us consider a uniform gas cylinder of radius $r=R_0$. The cylinder is immersed in an infinite liquid of density ρ . The liquid has a basic flow of uniform velocity U along the axis of the hollow cylinder. The model (hollow jet) is acting upon the capillary (along the gas-liquid interface) and inertia forces such that the liquid inertia force is paramount [3] over that of the gas . The effect of gravity

is negligible for $(gR_0/U^2) \ll 1$. We shall use the cylindrical polar coordinates (r, ρ, z) with the axis of the hollow cylinder coinciding with the z -axis. The fluids are considered to be inviscid and incompressible. The basic equations are those of motion, continuity and that one of surface tension viz [1]

$$(1) \quad \rho \frac{d\mathbf{u}}{dt} = \rho \left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} \right) = -\nabla p$$

$$(2) \quad \text{div } \mathbf{u} = 0$$

$$(3) \quad P_s = -T \left(\frac{1}{r_1} + \frac{1}{r_2} \right) = -T \text{div } \mathbf{n}_s,$$

where $\mathbf{u}$, P and T are the liquid velocity, pressure and surface tension coefficient: r_1 and r_2 are principle radii of curvature of the cylindrical surface and $\mathbf{n}_s (= \nabla f / |\nabla f|)$ is a unit outward vector normal (directed from the gas to the liquid) to the surface $f(r, \rho, z; t) = 0$.

The equilibrium pressure P_0 is given by

$$(4) \quad P_0 = P_g - (T/R_0).$$

The term $(-T/R_0)$ is the surface tension contribution and P_g is a gas constant pressure. The hollow cylinder has to be filled with gas of strong (constant) pressure to insure the equilibrium state such that $P_g \geq T/R_0$ (where equality may occur as limiting case with zero liquid pressure) otherwise the model will collapse towards a hollow cylinder of smaller radius than R_0 to reach an equilibrium state.

For small departures from the equilibrium state, let the small departure in the radius of the cylinder be ζ and proportional to $\exp[i(\alpha z + m\rho) - i\alpha c t]$ where $\zeta \ll R_0$.

$$(5) \quad r = R_0 + \zeta = R_0 + \varepsilon_0 R_0 \exp[i(\alpha z + m\rho) - i\alpha c t]$$

where ϵ_0 is the initial amplitude (at time $t = 0$); α and m are the longitudinal and azimuthal wavenumbers, c is the velocity of light. ζ represented the surface wave elevation normalized with respect to R_0 and measured from the equilibrium position. The linearized equations are

$$(6) \quad \rho \left(\frac{\partial u_1}{\partial t} + U \frac{\partial u_1}{\partial z} \right) = - \nabla P_1$$

$$(7) \quad \text{div } u_1 = 0$$

$$(8) \quad P_{1s} = (T/R_0^2) \left(\zeta = \left(\partial^2 \zeta / \partial \rho^2 \right) + R_0^2 \left(\partial^2 \zeta / \partial z^2 \right) \right) : \text{ at } r = R_0$$

Consequent to deformation (5) as usual for the stability problems of cylindrical configurations, based on the linear-perturbation technique we assume that any perturbed quantity can be expressed as $[i(\alpha z + m\phi) - i\alpha c t]$ times an amplitude function of r . The relevant perturbation equations are solved and the non-singular solution is given by

$$(9) \quad u_1 = A \left[\alpha K'_m(\alpha r), \frac{im}{r} K_m(\alpha r), i\alpha K_m(\alpha r) \right]$$

$$\exp [i(\alpha z + m\phi) - i\alpha c t]$$

$$(10) \quad P_1 = -\rho (U - c) u_{1z}$$

$$(11) \quad \epsilon_0^{-1} P_{1s} = (T/R_0) (1 - m^2 - \alpha^2 R_0^2) \exp [i(\alpha z + m\phi) - i\alpha c t],$$

where A is unspecified constant and $K_m(\alpha r)$ is the modified Bessel function of the second kind of the order m . The appropriate boundary conditions are: (I) the kinematic condition that the normal component of the fluid velocity must be compatible with the deformed

surface (5) at $r = R_0$, yields $u_{1r} = \partial \zeta / \partial t + U \partial \zeta / \partial z$ from which one gets $A = iR_0 (U - c) / K'_m (\alpha R_0)$; and (II) the kinetic pressure of the fluid must cope with the pressure due to the capillary force at $r = R_0$. The last condition gives, at once, the following relation :

$$(12) \quad \omega^2 = (T/\rho R_0^3) (x^2 + m^2 - 1) F_m(x)$$

where $\omega = (U - c) \alpha$

$$F_m(x) = -x K'_m(x) / K_m(x)$$

and where $x (= \alpha R_0)$ is the longitudinal dimensionless wavenumber.

3. DISCUSSIONS AND CONCLUSIONS

Equation (12) is the desired eigenvalue relation of a capillary streaming hollow cylinder (valid for all modes of perturbation $m \geq 0$) subjected to its liquid inertia force. It relates the angular frequency ω (or rather the temporal amplification σ if ω is imaginary ($\sigma = -iac$)) with the entity $(T/\rho R_0^3)^{-1/2}$ as unit of time, the wave-number x and m , the second kind modified Bessel function and its derivative (with respect to its argument), the characteristic length R_0 and the other parameters T , ρ , U and c of the problem.

In the absence of streaming ($U = 0$) and to axisymmetric disturbances $m = 0$; equation (12) reduces to

$$(13) \quad \sigma^2 = (T/\rho R_0^2) [x K_1(x) / K_0(x)] (x^2 - 1).$$

This coincides with dispersion relation obtained by us [7] as the viscosity effect is suppressed there.

By using the recurrence relation of the modified Bessel function

$$(14) \quad 2 K'_m(x) = -K_{m-1}(x) - K_{m+1}(x)$$

and to each non-zero real value of x such that $K_m(x)$ is monotonic decreasing but never negative. One can show that

$$(15) \quad F_m(x) > 0$$

to all modes of perturbation $m \geq 0$ for all (long and short) wavelengths and that $F_m(x)$ never changes sign. By using the inequality (15); equation (12) yields that $\omega^2 > 0$ for all $m \neq 0$, but that $\omega^2 < 0$ for $-1 < x < 1$ and $\omega^2 \geq 0$ for $x \geq 1$ or $x \leq -1$ if $m = 0$. That is exactly the same situation for the nonhollow cylinder (full liquid jet). It is stable to all nonaxisymmetric disturbances $m \geq 1$ for all wavelengths while to $m=0$ it is stable (or not) if the perturbed wavelength is equal to or shorter (or longer) than the circumference of the jet. From the analytical discussions of eq. (12), one can easily show that the streaming has a destabilizing effect. The latter attains its maximum value if the basic flow streams (for given U) in the opposite z -direction. The destabilizing character of the streaming is also true even in the presence of different forces [9]. The analytical results are confirmed numerically by using the relation (12) and that of full liquid jet

$$(6) \quad \omega^2 = (T / \rho R_0^3) [x I'_m(x) / I_m(x)] (\lambda^2 + m^2 - 1)$$

(where ω , T , ρ , R_0 , x and m have their usual meaning while $I_m(x)$ is the first kind modified Bessel function of the order m) in the computer simulation to axisymmetric mode $m = 0$ and non-axisymmetric modes 1 and 2.

The oscillation curves of the hollow cylinder (for all x values to $m=1$ and 2 and also to $m=0$ if $x \geq 1$) are little bit higher than those of the full jet in the same domains of stability .

The domain of instability is $0 < x < 1$ to $m=0$ only . The maximum temporal amplification prevailing the nonhollow case is for lower than that of the present model . The maximum modes of instability ($=0.34$ for full jet and $=0.82$ for hollow cylinder) give fair feeling for that . This may be due to the fact that the pressure pushes in the same direction as the surface tension does. This reason is not only the cause of that since we have used volume conservation and the cylinder is filled with gas of (constant) strong equilibrium pressure . However, another reason can be stated if one takes into account the conservation energy principle . The only source that can deliver energy is the potential energy due to surface tension . The greater instability is due to the fact that the same amount of potential energy can more easily give rise to motions, i. e., increasing the kinetic energy according to the energy conservation principle (needing little kinetic energy) because the hollow cylinder extends radially to infinity ; in comparison with the liquid cylinder (which) due to its radial limitation needs comparatively large speeds in the z -direction in particular for long wavelengths $x \ll 1$) uses more kinetic energy for the same radial deformations .

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PERTURBATIONS OF PERIGEE AND APOGEE DISTANCE OF EARTH'S SATELLITE ORBITS DUE TO EARTH'S OBLATENESS AND SOLAR RADIATION PRESSURE

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1. INTRODUCTION

Levin [1] studied the solar radiation pressure perturbation effect on the satellite orbit and calculated apsidal locations and distances, neglecting the other perturbing forces such as Earth's oblateness. He calculated all his results by taking the sun and the satellite orbit in the equatorial plane and ignored the obliquity of the ecliptic. In this paper we have studied the distances and locations of the perigee and apogee by taking into account the effect of solar radiation pressure and oblateness of the Earth. We have taken the sun and the satellite orbit in the plane of ecliptic. We have considered the problem for a fixed source as well as moving source. Shadow effect of the Earth is also taken into account. In all the cases, the orbit of the satellite is supposed to be circular but in section 4, it is taken to be elliptic. However the eccentricity considered is of the $O(10^{-3})$. We have taken the perturbing acceleration upto $O(10^{-5})$.

2. EQUATIONS OF MOTION

The force exerted by radiation pressure upon a satellite depends on the intensity of the radiation, the presented normal area of the satellite and reflectivity of the surface. The magnitude of the perturbing acceleration on a particle of mass m and cross-section A_N may be written as Levin [1]

$$K = \frac{(1+\rho)s}{C} \times \frac{A_N}{m} \text{ cm/sec}^2$$

where ρ is the reflectivity, $0 \leq \rho \leq 1$, s is the solar constant ($\sim 1.35 \times 10^6 \text{ ergs/cm}^2 \text{ sec}$) and C is the velocity of light ($\sim 3 \times 10^{10} \text{ cm/sec}$).

For the sake of simplicity, we assume a nominally circular orbit of the satellite and fixed sun at $x = -\infty$ anywhere in the ecliptic plane. We also assume that the Earth, the sun and the satellite orbit are in the same plane. Further, we neglect the shadow and the reflected radiation by the earth. If X is the position of the satellite which moves in a circular orbit around the Earth (E) Fig. 1 (a, b) and the

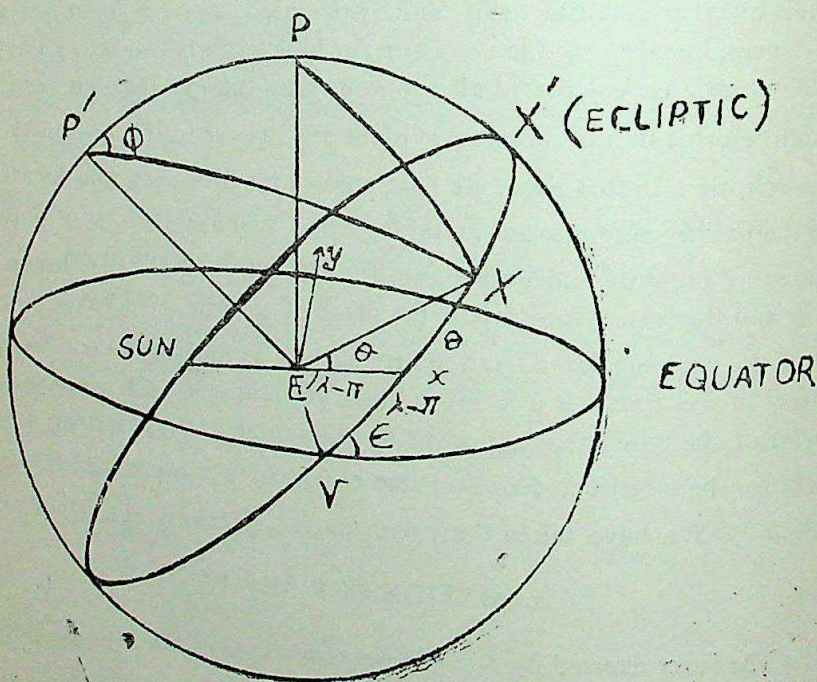
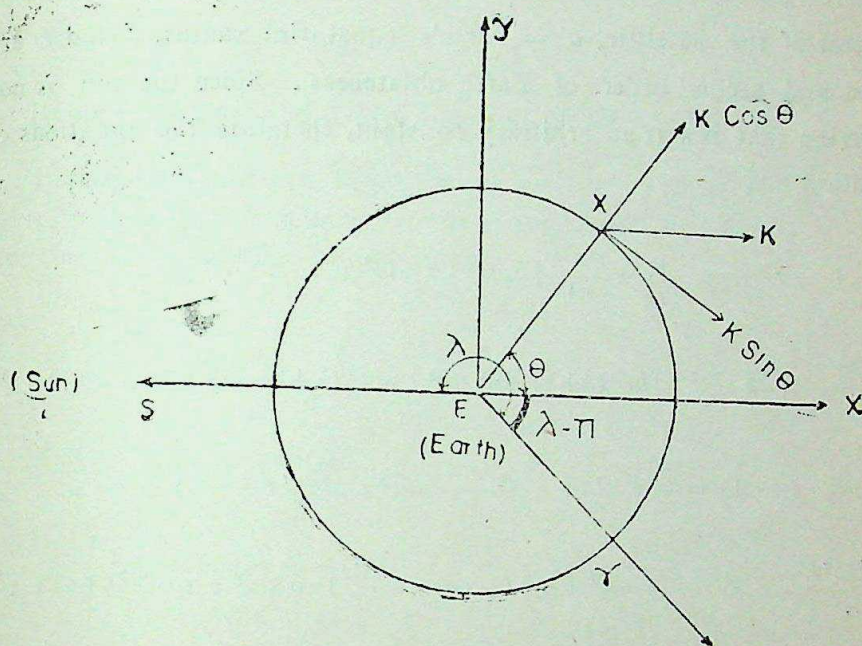


FIG 1 (a)



(Fig. 1)(b)

angle θ that EX makes with SE produced, the line joining the Sun and the Earth, then the radial and transverse components of the perturbing acceleration are $K_r = K \cos \theta$; $K_\theta = -K \sin \theta$. If λ is Sun's longitude and δ the declination of the satellite, we have $\sin \delta = -\sin \epsilon \sin (\lambda + \theta)$ where ϵ is the obliquity.

Again, the potential of the Earth at an external point is given by Sterne [3]

$$V = \frac{\mu}{r} \left[1 - \frac{1}{3} \frac{Ja'^2}{r^2} (3 \sin^2 \delta - 1) + \frac{D}{35} \left(\frac{a'}{r} \right)^4 (35 \sin^4 \delta - 30 \sin^2 \delta + 3) \right]$$

where $\mu = Mf$; M = mass of the Earth; f = gravity constant, r = radius vector of the satellite, a' = Earth's equatorial radius, J and D are first and second order of Earth oblateness. Since the sun is not moving that is λ is an arbitrary constant, therefore the equations of motion are

$$\ddot{r} - r \dot{\theta}^2 = -\frac{\mu}{r^2} + \frac{Ja'^2}{r^4} [3 \sin^2 \epsilon \sin^2 (\lambda + \theta) - 1] -$$

$$\frac{\mu Da'^4}{7r^6} [35 \sin^4 \epsilon \sin^4 (\lambda + \theta) - 30 \sin^2 \epsilon \sin^2 (\lambda + \theta) + 3] + K \cos \theta \quad (1)$$

$$\frac{1}{r} \frac{d}{dt} (r^2 \dot{\theta}) = -K \sin \theta - \frac{\mu Ja'^2}{r^4} \sin^2 \epsilon \sin 2(\lambda + \theta) +$$

$$\frac{\mu Da'^4}{7r^6} \{ (28 \sin^4 \epsilon \sin^3 (\lambda + \theta) \cos (\lambda + \theta) - 6 \sin^2 \epsilon \sin 2(\lambda + \theta)) \} \quad (2)$$

We take the initial conditions as

$$\theta_0 = -\pi/2; r = r_0; (r^2 \dot{\theta})_0^2 = \mu r_0; \dot{r} = 0.$$

Putting $P = r^2 \dot{\theta}$ in equation (2), integrating and applying initial conditions, we get P^2 . If we put $u = 1/r$ and using the value of P^2 , we get a second order differential equation. We have included terms $O(10^{-5})$. Solving this differential equation applying initial conditions and substituting $u = 1/r$, we get

$$r = a + f(\sin \theta, \cos \theta) + A\theta \sin \theta - B\theta \cos \theta \quad (3)$$

$$\text{where } A = \frac{KJr_0 a'^2}{\mu} \left\{ -3 \sin^2 \epsilon \cos 2\lambda - \frac{3}{2} \sin^2 \epsilon + 1 \right\} + \frac{3Kr_0^3}{2\mu} > 0, \quad \left. \begin{array}{l} \dots (3a) \\ \forall \lambda, \text{ as } \sin \epsilon \simeq \frac{2}{5} \end{array} \right\}$$

$$B = \frac{KJa'^2 r_0}{2\mu} \sin^2 \varepsilon \sin 2\lambda,$$

'a' is some constant and f is a function of $\sin \theta$ and $\cos \theta$.

Finally, we find the approximate solution as $\theta_{apsc} = n\pi + \alpha$, where n is any positive integer or zero and $\alpha = \tan^{-1}(-A/B)$.

Therefore from equation (3), we have

$$\begin{aligned} r_{apsc} = r_0 & - \left[\frac{1}{3} \frac{Ja'^2}{r_0} \sin^2 \varepsilon \sin 2\lambda - \frac{3Kr_0^3}{2\mu} + \right. \\ & + \frac{KJr_0 a'^2}{2\mu} \sin^2 \varepsilon \{ (3 + 5 \cos 2\lambda) - 2 \} \\ & - \frac{KJr_0 a'^2}{8\mu} \sin^2 \varepsilon \{ 3 \cos 3\lambda - 4 \cos 2\lambda + 2(-\pi + 4\lambda + \sin 2\lambda) \} + \frac{6}{7} \frac{Da'^4}{r_0^3} \times \\ & \times \sin^2 \varepsilon \sin 2\lambda - \frac{1}{10} \frac{Da'^4}{r_0^3} \sin^4 \varepsilon \{ (\sin 4\lambda + 10 \sin 2\lambda) \\ & - \frac{2}{3} \frac{J^2 a'^4}{r_0^3} \sin^2 \varepsilon \sin 2\lambda + \frac{J^2 a'^4 \sin^4 \varepsilon}{r_0^3} \sin 2\lambda + \\ & + \frac{7 J^2 a'^4 \sin^4 \varepsilon}{30 r_0^3} \sin 4\lambda \} \cos(n\pi + \alpha) - \\ & - \left[\frac{Ja'^2}{r_0} \left\{ \left(1 - \frac{3}{2} \sin^2 \varepsilon - \frac{5}{6} \cos 2\lambda \sin^2 \varepsilon \right) + \right. \right. \\ & + \frac{Da'^4}{7r_0^3} \{ 14 \sin^4 \varepsilon \cos^4 \lambda + 3 - 3 \sin^2 \varepsilon (\cos 2\lambda + 5) \} + \\ & + \left. \left. \frac{Da'^4 \sin^4 \varepsilon}{40r_0^3} \{ -\cos 4\lambda - 20 \cos 2\lambda + 45 \} - \right. \right. \end{aligned}$$

$$\begin{aligned}
& - \frac{3Kr_0^3}{2\mu} \cdot \frac{\pi}{2} + \frac{KJr_0a'^2}{4\mu} \left\{ 6\pi \sin^2 \varepsilon \cos 2\lambda + \sin^2 \varepsilon \left(3\pi + \frac{\sin 2\lambda}{2} - \right. \right. \\
& \left. \left. - 4\lambda \cos 2\lambda \right) - 2\pi \right\} + \frac{4K^2 r_0^5}{\mu^2} - \frac{4}{3} \frac{J^2 a'^4 \sin^2 \varepsilon \cos 2\lambda}{r_0^3} + \\
& + \frac{J^2 a'^4 \sin^4 \varepsilon \cos^2 2\lambda}{r_0^3} + \frac{2J^2 a'^4 \sin^4 \varepsilon \cos 2\lambda}{r_0^3} + \frac{1}{10} \frac{J^2 a'^4 \sin^4 \varepsilon \cos 4\lambda}{r_0^3} + \\
& + \frac{J^2 a'^4 \sin^4 \varepsilon}{12r_0^3} \left[\sin(n\pi + \alpha) - \frac{Ja'^2}{r_0} \left\{ 1 - \left(\frac{3}{2} + \cos 2\lambda \right) \sin^2 \varepsilon \right\} - \right. \\
& \left. - \frac{Da'^4}{7r_0^3} \left\{ 14 \sin^4 \varepsilon \cos^4 \lambda + 3 - 3 \sin^2 \varepsilon (2 \cos 2\lambda + 5) \right\} + \right. \\
& \left. + \frac{J^2 a'^4 \sin^2 \varepsilon \cos 2\lambda}{r_0^3} - \frac{3J^2 a'^4 \sin^4 \varepsilon \cos 2\lambda}{2r_0^3} - \frac{J^2 a'^4 \sin^4 \varepsilon \cos^2 2\lambda}{r_0^3} + \right. \\
& \left. + \frac{3Kr_0^3}{2\mu} (n\pi + \alpha) \sin(n\pi + \alpha) + \frac{1}{6} \frac{Ja'^2}{r_0} \sin^2 \varepsilon \cos(2n\pi + 2\alpha + 2\lambda) \right. \\
& \left. - \frac{KJa'^2 r_0}{2\mu} \left\{ \sin^2 \varepsilon (3 + 5 \cos 2\lambda - 2) \right\} (n\pi + \alpha) \sin(n\pi + \alpha) - \right. \\
& \left. - \frac{KJr_0a'^2}{8\mu} \sin^2 \varepsilon \{-\cos(3n\pi + 3\alpha + 2\lambda) + 4(n\pi + \alpha + 2\lambda) \sin(n\pi + \alpha + 2\lambda)\} \right. \\
& \left. - \frac{K^2 r_0^5}{\mu^2} \left\{ 3 - \cos(2n\pi + 2\alpha) \right\} + \frac{3}{7} \frac{Da'^4}{r_0^3} \sin^2 \varepsilon \cos(2n\pi + 2\alpha + 2\lambda) - \right.
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{40} \frac{Da'^4}{r_0^3} \sin^4 \varepsilon \{ -\cos 4(n\pi + \alpha + \lambda) + 20 \cos(2n\pi + 2\alpha + 2\lambda) + 45 \} - \\
& - \frac{J^2 a'^4 \sin^2 \varepsilon}{3r_0^3} \cos(2n\pi + 2\alpha + 2\lambda) + \frac{J^2 a'^4 \sin^4 \varepsilon}{12r_0^3} \cos(2n\pi + 2\alpha + 4\lambda) + \\
& - \frac{J^2 a'^4 \sin^4 \varepsilon}{12r_0^3} \cos(2n\pi + 2\alpha) + \frac{J^2 a'^4 \sin^4 \varepsilon}{2r_0^3} \cos(2n\pi + 2\alpha + 2\lambda) - \\
& - \frac{J^2 a'^4 \sin^4 \varepsilon}{60r_0^3} \cos(4n\pi + 4\alpha + 4\lambda).
\end{aligned}$$

Now in order to study the perigee and apogee distances, we shall consider the following two cases : (i) $B \neq 0$ and (ii) $B = 0$, where $\tan \alpha = -A/B$ and $A > 0$.

Case (i) (a) Suppose $B > 0$, this implies $0 < \lambda < \frac{\pi}{2}$ or $\pi < \lambda < \frac{3\pi}{2}$.

As $A > 0$, so $\tan \alpha = -A/B < 0$; $\frac{\pi}{2} < \alpha < \pi$. We get perigee or apogee according as n is odd or even :

$$r_p = r_{\text{perigee}} = a + f \{ \sin \alpha, \cos \alpha \} - (n\pi + \alpha) (A^2 + B^2)^{1/2}$$

$$r_a = r_{\text{apogee}} = a + f \{ \sin \alpha, \cos \alpha \} + (n\pi + \alpha) (A^2 + B^2)^{1/2}$$

(b) Suppose $B < 0$, this implies $\frac{\pi}{2} < \lambda < \pi$ or $\frac{3\pi}{2} < \lambda < 2\pi$. As $A > 0$,

so $\tan \alpha = -\frac{A}{B} > 0$; $0 < \alpha < \frac{\pi}{2}$. Again we get perigee or apogee

according as n is odd or even :

$$r_p = a + f \{ \sin \alpha, \cos \alpha \} - (n\pi + \alpha) (A^2 + B^2)^{1/2}.$$

Similarly apogee can be calculated .

Case (ii) (a) Suppose $\varepsilon \neq 0$, $\lambda = \pi \Rightarrow B=0$, therefore $\theta = n\pi - \frac{\pi}{2}$,

that is the case when sun is fixed in the line of vernal equinox at $x = -\infty$. We get perigee or apogee according as n is even or odd as

$$r_p = r_0 - \frac{Kr_0}{2\mu} (3r_0^2 + 2Ja'^2) \times 2m\pi + \frac{KJr_0a'^2}{\mu} \cdot \frac{9}{2} \sin^2 \varepsilon \times 2m\pi$$

Similarly apogee distance can be calculated .

(b) Suppose $\varepsilon=0$ then $B=0$ and there $\theta = n\pi - \pi/2$ that is, the case when the Sun is fixed anywhere in the equatorial plane at $x = -\infty$. We get perigee or apogee according as n is even or odd as

$$r_p = r_0 - \frac{Kr_0}{2\mu} (3r_0^2 + 2Ja'^2) \times 2m\pi, \text{ where } m \text{ is zero or positive integer .}$$

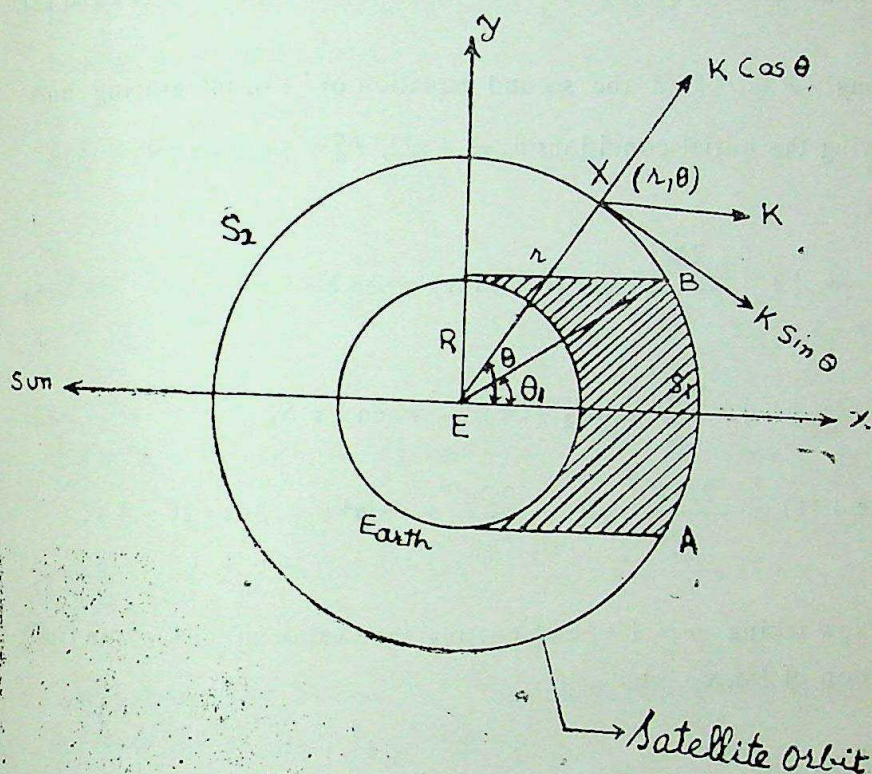
Similarly apogee can be calculated .

In both the above cases, we note that the rate of decrease of perigee and the rate of increase of apogee with each revolution is the same. therefore the semi-major axis $\frac{1}{2}(\text{perigee} + \text{apogee})$ remains fixed and therefore period of revolution is approximately constant .

3. EFFECT OF SHADOW

Again, we shall assume that the satellite moves in a circle and the radiation source is fixed as $x = -\infty$. We further assume that the Earth, the Sun and the satellite orbit are in the same plane .

Let the shadow region be defined by (Fig. 2)



(FIG - 2)

$$S_1: -\theta_1 \leq \theta \leq \theta_1; S_2: \theta_1 < \theta < 2\pi - \theta_1$$

where $\theta_1 = \cos^{-1} \sqrt{1 - \sigma^2}$, $R/r = \sigma$ and $R = \text{the radius of the Earth}$;

Equations of motion are

$$\left. \begin{aligned} \ddot{r} - r \dot{\theta}^2 &= F_r + K \cos \theta \\ \frac{1}{r} \frac{d}{dt} (r^2 \dot{\theta}) &= F_\theta - K \sin \theta \end{aligned} \right\} \begin{aligned} & \\ & \end{aligned} \left. \begin{aligned} & \\ & \end{aligned} \right\} \begin{aligned} & \\ & \end{aligned} \quad (4)$$

where the values of F_r and F_θ can be written from equations (1) and (2).

Putting $P = r^2 \dot{\theta}$ in the second equation of (4), integrating and applying the initial conditions $\theta_0 = -\pi/2$, $P_0^2 = \mu r_0$, we get

$$P^2 = \mu r_0 \left[1 + \frac{2Kr_0^2}{\mu} \cos \theta + \phi(\theta) \right] \text{ when } \theta \in S_2 \quad (5)$$

$$P^2 = \mu r_0 \left[1 + \frac{2Kr_0^2}{\mu} \cos \theta_1 + \phi(\theta) \right] \text{ when } \theta \in S_1 \quad (6)$$

$$\text{where } \phi(\theta) = \frac{2Ja'^2}{r_0^2} \cos^2 \theta + \frac{2Da'^2}{7r_0^4} (7 \sin^4 \theta + 3 \cos 2\theta - 4).$$

Now taking $u = 1/r$ and putting the value of P^2 in the first equation of (4), we get

$$\frac{d^2u}{d\theta^2} + u = \frac{1}{r_0} - f(\theta) \quad (7)$$

$$\text{where } f(\theta) = \frac{3Kr_0}{\mu} \cos \theta - \frac{4KJa'^2}{\mu r_0} (1 + \cos 2\theta) \cos \theta -$$

$$- \frac{6K^2r_0^3}{\mu^2} \cos^2 \theta + P(\theta), \theta \in S_2$$

and similarly the value of $f(\theta)$ can be calculated when $\theta \in S_1$,

$$\text{where } P(\theta) = \frac{Ja'^2}{r_0^3} (1 + \sin^2 \theta) - \frac{Da'^4}{7r_0^5} (21 \sin^4 \theta - 24 \sin^2 \theta + 8) -$$

$$- \frac{2J^2a'^4}{r_0^5} \cos^2 \theta (2 - \cos^2 \theta).$$

In order to find out the solution of (7), $f(\theta)$ may be expressed by its Fourier series in the form

$$f(\theta) = A_0 + A_1 \cos \theta + \sum_{m=2}^{\infty} A_m \cos m\theta$$

$$\text{where } A_0 = \frac{Kr_0}{\mu\pi} \{ 2\theta_1 \cos \theta_1 - 3 \sin \theta_1 \} + \frac{3Ja'^2}{2r_0^3} + \frac{KJa'^2}{\pi\mu r_0} (14 \sin \theta_1 +$$

$$\frac{2}{3} \sin 3\theta_1 - 5\theta_1 \cos \theta_1 - \sin \theta_1 \cos^2 \theta_1) - \frac{K^2 r_0^3}{\pi \mu^2} \{ 3(\pi - \theta_1) - \cos \theta_1 (3 \sin \theta_1 - 4\theta_1 \cos \theta_1) \} - \frac{31}{56} \frac{Da'^4}{r_0^5} - \frac{5}{4} \frac{J^2 a'^4}{r_0^5} \quad (8)$$

Similarly A_1 and A_m can easily be calculated. Solving equation (7) applying initial conditions, we get

$$u = \frac{1}{r_0} - A_0 - \left(\frac{A_1}{2} + S_0 \right) \cos \theta - (A_0 - S_1) \sin \theta - \frac{A_1}{2} (\theta + \pi/2) \sin \theta + \sum_{m=2}^{\infty} \frac{A_m \cos m\theta}{(m^2 - 1)} \quad (9)$$

$$\text{where } S_0 = \sum_{m=1}^{\infty} \frac{(2m+1)(-1)^{m+1} A_{2m+1}}{(2m+1)^2 - 1}, S_1 = \sum_{m=1}^{\infty} \frac{(-1)^m A_{2m}}{4m^2 - 1}$$

The solution of (9) is $\theta_{apse} \simeq 2n\pi \pm \pi/2$. From equation (9), we have

$$r_{perigee} = r_0 \left[1 - 2n\pi \cdot \frac{3Kr_0^2}{2} \left\{ 1 - \frac{\theta_1}{\pi} + \frac{\sin 2\theta_1}{6\pi} \right\} + 2n\pi \cdot \frac{7KJa'^2}{2\mu} \right] \times$$

$$\left\{ 1 - \frac{\theta_1}{\pi} - \frac{4}{7\pi} \sin 2\theta_1 - \frac{1}{28\pi} \sin 4\theta_1 + \frac{\cos \theta_1}{7\pi} (11 \sin \theta_1 + \frac{1}{3} \sin 3\theta_1) \right\} - 2n\pi \cdot \frac{K^2 r_0^4}{2\pi\mu^2} \{ 9 \sin \theta_1 + \sin 3\theta_1 - 8 \cos^2 \theta_1 \sin \theta_1 \} \quad (10)$$

$$\text{and } r_{\text{apogee}} = r_0 (1 + 2r_0 (A_0 \cdot S_1) + \frac{A_1 r_0}{2} (2n\pi + \pi)) .$$

For a low to medium altitude satellite, θ_1 is large and therefore $\frac{\theta_1}{\pi}$ is the dominant term in (10). Now as $\frac{\theta_1}{\pi}$ is the portion of the circumference in shadow, we note that the effect of shadow is to diminish the height of the perigee by an amount equal to the fraction of the path which is in shadow. Further, for a very distant satellite $\theta_1 \rightarrow 0$ and therefore the effect of shadow is almost negligible.

4. PERIGEE AND APOGEE DISTANCES FOR ELLIPTIC ORBIT

In this section, we shall try to find apsidal distances and their angular positions for elliptic case taking the radiation source as fixed. The eccentricity considered is of the $O(10^{-3})$.

The equations of motion are the same as given in (4). Putting $P = r^2 \dot{\theta}$ in these equations and using $r = \frac{l}{1 + e \cos(\theta - \Psi)}$, where Ψ is the initial angular position of perigee and $l = a(1 - e^2) = P_0^2/\mu$ is the semi-latus rectum of the ellipse and after applying initial conditions $\theta_0 = -\pi/2$, $P^2 = P_0^2 = \mu l$; we get the value of P^2 . As before let $u_0 = 1/r$ and putting the value of P^2 in (4) and solving it and applying initial conditions $\theta_0 = -\pi/2$, $u = u_0 = \{1 - e \sin \Psi\} / a(1 - e^2)$; $(du/d\theta)_0 = (e \cos \Psi) / a(1 - e^2)$, we obtain

$$\begin{aligned}
u = & \frac{1+e \cos (\theta-\Psi)}{l} - \frac{3Kl}{2\mu} \{ \cos \theta + (\theta + \pi/2) \sin \theta \} - \frac{3Kle}{2\mu} \{ (\pi \sin \Psi - \cos \Psi) + \\
& + 2\theta \sin \Psi + \frac{1}{3} \cos (2\theta - \Psi) - \frac{4}{3} \sin (\theta + \Psi) \} - \frac{Ja'^2}{6l^3} (8 \sin \theta + \cos 2\theta + 9) + \\
& + \frac{Da'^4}{280l^3} (128 \sin \theta - 7 \cos 4\theta + 20 \cos 2\theta + 155) + \frac{KJa'^2}{8\mu l} (25 \cos \theta - \cos 3\theta + \\
& + 28 (\pi/2 + \theta) \sin \theta) + \frac{K^2 l^3}{\mu^2} (4 \sin \theta - \cos 2\theta + 3) \quad (11)
\end{aligned}$$

We get $\theta_{apsc} \simeq 2n\pi \pm \pi/2$. From equation (11), we get the perigee distance as

$$V_{perigee} = a (1 + e \sin \Psi) - \frac{3Ka_3}{\mu} (\frac{1}{2} - e \sin \Psi) 2n\pi + \frac{7KJa'^2}{2\mu} \cdot 2n\pi a.$$

Similarly, we can obtain apogee by putting $\theta = 2n\pi + \pi/2$ in equation (11).

However, perigee is initially located at $\theta = \Psi$, but for small eccentricity after a moderate number of revolutions, it is shifted from Ψ to $(-\pi/2 + 2n\pi)$. And if the eccentricity is large the motion is much slower and the initial perigee location tends to persist.

5. MOVING SOURCE

If the Sun moves from its initial direction at a constant angular rate N' , then the radial and transverse components of the perturbing acceleration due to solar pressure at any time t are

$$K_r = K \cos (\theta - N't) ; K_\theta = -K \sin (\theta - N't).$$

As before, we can write the equations of motion and putting $r^2 \dot{\theta} = P$ in the second equation, integrating and taking the initial conditions

as $\theta_0 = -\pi/2$, $r=r_0$, $(r^2 \dot{\theta})_0^2 = \mu r_0$, $\dot{r} = 0$, we get the value of P^2 .

We note that $\dot{\theta} = \left(\frac{\mu}{r_0^3}\right)^{\frac{1}{2}} = \text{constant} = N$ (say), then
 $Nt = \theta + \pi/2$.

Also, $N't = N'/N (Nt) = N'/N (\theta + \pi/2)$.

Now if we put $u=1/r$ in the first equation of motion and use the value of P^2 , we get a second order differential equation. solving it and taking into account that $N'/N \ll 1$, we have

$$U \simeq C + f(\sin \theta, \cos \theta) + \lambda_3 \cos(3\theta - N't) + \lambda_4 \cos 2(\theta - N't) + N/N' \{ -(\lambda_1 + \lambda_2) \cos \theta + \lambda_1 \cos(\theta - N't) + \lambda_2 \cos(\theta + N't) \} \quad (12)$$

where

$$C \simeq \frac{1}{r_0} + \frac{Ja'^2}{r_0^3} \left(1 - \frac{5}{2} \sin^2 \epsilon\right) + \frac{Da'^4}{7r_0^5} (14 \sin^4 \epsilon - 21 \sin^2 \epsilon + 3) +$$

$$\frac{9}{8} \frac{Da'^4}{r_0^5} \sin^4 \epsilon + \frac{3K^2 r_0^3}{\mu^2} + \frac{J^2 a'^4 \sin^2 \epsilon}{2r_0^5} \left(\frac{9}{4} \sin^2 \epsilon - 2\right),$$

$$f(\sin \theta, \cos \theta) \simeq \lambda \cdot N/N' \cos \theta + \left\{ \frac{Ja'^2}{r_0^3} \left(1 - \frac{7}{3} \sin^2 \epsilon\right) + \right.$$

$$\frac{Da'^4}{35r_0^5} (91 \sin^4 \epsilon - 90 \sin^2 \epsilon + 15) + \frac{4K^2 r_0^3}{\mu^2} + \frac{J^2 a'^4 \sin^2 \epsilon}{120r_0^5} \\ \left. (217 \sin^2 \epsilon - 160) \right\} \sin \theta -$$

$$- \frac{Ja'^2 \sin^2 \epsilon}{6r_0^3} \cos 2\theta - \frac{3}{7} \cdot \frac{Da'^4}{r_0^5} \sin^2 \epsilon \cos 2\theta + \frac{Da'^4 \sin^4 \epsilon}{40r_0^5} (-\cos 4\theta +$$

$$+ 20 \cos 2\theta) - \frac{J^2 a'^4 \sin^2 \epsilon}{3r_0^5} (2 \sin^2 \epsilon - 1) \cos 2\theta + \frac{J^2 a'^4 \sin^4 \epsilon}{60r_0^5} \cos 4\theta,$$

$$\lambda_1 = -\frac{3Kr_0}{2\mu} + \frac{KJa'^2}{\mu r_0} (4 \sin^2 \epsilon - 1); \lambda_2 = -\frac{KJa'^2 \sin^2 \epsilon}{2\mu r_0},$$

$$\lambda_3 = -\frac{KJa'^2 \sin^2 \epsilon}{8\mu r_0}, \lambda_4 = -\frac{3K^2 r_0^3}{\mu 2}$$

$$\text{and } \lambda = \frac{3Kr_0}{2\mu} \cdot \frac{N}{N'} - \frac{KJa'^2 (4 \sin^2 \epsilon - 1)}{\mu r_0} \cdot \frac{N}{N'} + \frac{KJa'^2 \sin^2 \epsilon}{2\mu r_0} \cdot \frac{N}{N'}.$$

we find the approximate solution of equation (12) is

$$\tan \theta = -p \cot \frac{N't}{2} \text{ where } p = \frac{(\lambda_1 - \lambda_2)}{(\lambda_1 + \lambda_3)} > 0 \quad (13)$$

Taking $\sin \epsilon \simeq \frac{2}{5}$, we have $P \simeq (1 - \beta)$, where $\beta = \frac{8}{75} \cdot \frac{Ja'^2}{r^2}$. Here,

it is to be noted that P depends upon oblateness and β a small quantity ($\beta \ll 1$). Also when $J = 0$, $\beta = 0$ and therefore $P = 1$, the solution

(13) becomes $\theta = n\pi + \frac{\pi}{2} + \frac{N't}{2}$. This confirms the findings of

Levin [1]. Now taking the solution (13), using $u = 1/r$ and taking

into account that $\frac{N}{N'} \ll 1$, we obtain

$$r_{\text{apse}} \simeq r_0 \left[1 \mp \frac{3Kr_0^2}{\mu} \frac{N}{N'} \sin \frac{N't}{2} (1 + \beta \cos^2 \frac{N't}{2} \pm \frac{Ja'^2}{r_0^2} (1 - \frac{7}{2} \sin^2 \epsilon)) \times \right. \\ \left. \times \cos \frac{N't}{2} (1 - \beta \sin^2 \frac{N't}{2}) - \frac{Ja'^2}{r_0^2} (1 - \frac{5}{2} \sin^2 \epsilon) + \right]$$

$$\begin{aligned}
& + \frac{Ja'^2 \sin^2 \epsilon}{6r_0^4} (\beta \sin^2 N't - \cos N't) \\
& \pm \frac{Da'^4}{35r_0^4} (91 \sin^4 \epsilon - 90 \sin^2 \epsilon + 15) \cos \frac{N't}{2} (1 - \beta \sin^2 \frac{N't}{2}) - \\
& - \frac{Da'^4}{56r_0^4} (175 \sin^4 \epsilon - 168 \sin^2 \epsilon + 24) + \frac{3Da'^4}{7r_0^4} \sin^2 \epsilon (\beta \sin^2 N't - \cos N't) \\
& - \frac{Da'^4 \sin^4 \epsilon}{40r_0^4} \{ -2 (\cos^2 N't - \beta \sin N't \sin 2N't) + 1 + \\
& \qquad \qquad \qquad 20 (\beta \sin^2 N't - \cos N't) \} - \\
& \frac{3K^2 r_0^4}{\mu^2} \mp \frac{3K^2 r_0^4}{\mu^2} \cos \frac{N't}{2} (1 - \beta \sin^2 \frac{N't}{2}) + \frac{3Kr_0^4}{\mu^2} (\beta \sin^2 N't + \cos N't) - \\
& \frac{J^2 a'^4 \sin^2 \epsilon}{2r_0^4} (\frac{9}{4} \sin^2 \epsilon - 2) + \frac{J^2 a'^4 \sin^2 \epsilon (2 \sin^2 \epsilon - 1)}{3r_0^4} (\beta \sin^2 N't - \cos N't) \\
& + \frac{J^2 a'^4 \sin^4 \epsilon}{60r_0^4} \{ 1 - 2 (\cos^2 N't - \beta \sin N't \sin N't) \mp \frac{KJa'^2 \sin^2 \epsilon}{8\mu} \\
& \times \{ \sin \frac{N't}{2} + 3\beta \cos \frac{N't}{2} \sin \frac{N't}{2} (\cos \frac{3N't}{2} + 2 \sin \frac{N't}{2} \sin N't) \} \\
& \mp \{ - \frac{KJa'^2 (4 \sin^2 \epsilon - 1)}{\mu} + \frac{KJa'^2 \beta}{\mu} (\frac{9}{2} \sin^2 \epsilon - 1) \cos^2 \frac{N't}{2} \sin \frac{N't}{2} \\
& + \frac{KJa'^2}{\mu} (1 - \frac{7}{2} \sin^2 \epsilon) \sin \frac{N't}{2} - \frac{KJa'^2 \sin^2 \epsilon}{2\mu} (\sin \frac{3N't}{2} + \\
& + \frac{\beta}{2} \sin N't \cos \frac{3N't}{2}) \}]
\end{aligned}$$

The upper sign corresponds to perigee and the lower sign to apogee. If we take terms only upto the order of $O(10^{-3})$, we have

$$r_{\text{perigee}} \approx r_0 [1 - r_s - r_E], \text{ where } r_s = \frac{3Kr_0^2}{\mu} \frac{N}{N'} \sin \frac{N't}{2},$$

$$r_E = \frac{4}{75} \frac{Ja'^2}{r_0^2} (1 - x) \left(\frac{43}{4} - x\right); x = \cos \frac{N't}{2} \ll 1, 0 \leq t < \infty,$$

Since $x < 1$, $r_E > 0$ for all values of t except when $x = \cos \frac{N't}{2} = 1$ or $\frac{43}{4}$. In this case $r_E = 0$. This means that the perigee distance is further decreased due to oblateness. Moreover r_E is a function of time. The greatest value is $94/75$, Ja'^2/r_0^2 which is attained after a complete revolution of the Sun that is when $N't = 2\pi$. In the second revolution, the value r_E starts decreasing and it becomes zero after the second complete revolution that is when $N't = 4\pi$. The terms r_s due to solar radiation pressure indicates that the perigee distance is decreased and its greatest value is $3Kr_0^2 / \mu \cdot \frac{N}{N'}$, which is obtained after $1/2$ revolution of the Sun. Both r_s and r_E are periodic functions with period T and $2T$ respectively. Thus we see that the effects of moving source and the oblateness of the Earth are to change the linear decay of perigee into a periodic fluctuation.

The above results tally with those of Levin when $\epsilon=0$ and $J=0$.

5. CONCLUSION

When the source is taken fixed, Levin proved that the perigee is fixed at $-\pi/2$, but the inclusion of oblateness results in the

shifting of perigee from $-\pi/2$ to $\pi + \alpha$ ($\alpha = \tan^{-1}(-A/B)$) A, B defined by equation (3a). The effect of oblateness is to decrease further the perigee distance or to increase the apogee distance. The effect of shadow is to diminish the height of the perigee by an amount equal to the fraction of the path which is in shadow and for a distant satellite, the effect of shadow is negligible. For small eccentricity the perigee, though initially located at $\theta = \Psi$, yet after a moderate number of revolutions is shifted from Ψ to $(-\pi/2 + 2n\pi)$ and for large eccentricity the motion is much slower and the initial perigee location tends to persist.

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SERIES OF SQUARES OF LEGENDRE OF THE SECOND KIND

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ABSTRACT

A general result of Bailey is employed to obtain new summation formulae involving squares of the Legendre function of the second kind . Closed - form expressions of certain generalised hypergeometric functions are used in the analysis .

1. INTRODUCTION

The author recently required a closed-form summation of a particular series of squares of the Legendre function of the second kind $Q_{\nu+n}(x)$. In spite of the importance of this function in potential theory, there seems to be a gap in the literature with regard to series of the type

$$\sum_{n=0}^{\infty} C_n [Q_{\nu+n}(x)]^2, \quad (1.1)$$

where C_n is a general coefficient. In order to tackle this problem, we note that

$$Q_{\nu+n}(x) = \sqrt{\pi} \Gamma(1+\nu+n) / \Gamma(3/2+\nu+n) [2(x-1)]^{-\nu-n-1} \\ \times {}_2F_1(1+\nu+n, 1+\nu+n; 2+2\nu+2n; \frac{2}{1-x}). \quad (1.2)$$

(See Erdélyi et al. [2, p. 132] .)

Slater [3, p. 80] has given the relation

$${}_2F_1(a, b; c; x) {}_2F_1(a, c-b; c; x) \\ = (1-x)^{-a} {}_4F_3(a, b, c-a, c-b; c, c/2, c/2+1/2; \frac{x^2}{4(x-1)}), \quad (1.3)$$

and if the parameters are suitably specialised, we have

$$[Q_{v+n}(x)]^2 = \frac{\pi \Gamma(1+v+n) \Gamma(1+v+n)}{\Gamma(3/2+v+n) \Gamma(3/2+v+n)} [4(x^2-1)]^{-v+n-1} \\ \times {}_3F_2(1+v+n, 1+v+n, 1+v+n; 2+2v+2n, 3/2+v+n; \frac{1}{1-x^2}), \quad (1.4)$$

which is a convenient result for the present purpose,

2. A GENERAL RESULT

W. N. Bailey [1] gave the following important result on the manipulation of series :

$$\text{If } \beta_n = \sum_{r=0}^n \alpha_r u_{n-r} v_{n+r} \quad (2.1)$$

$$\gamma_n = \sum_{r=n}^{\infty} \delta_r u_{r-n} v_{r+n}, \quad (2.2)$$

where α_r , δ_r , u_r and v_r are functions of r only, such that the series γ_n exists, then

$$\sum_{n=0}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \beta_n \delta_n, \quad (2.3)$$

provided that all the series concerned are either convergent or terminating.

If we let

$$u_r = 1/r!, v_r = 1/(2+2v, r), \delta_r = [(1+v, r)]^3 / (3/2+v, r) (1-x^2)^{-r}$$

and

$$a_r = \frac{((a), r) y^r}{((h), r) r!}, \quad (2.4)$$

where

$$(a, r) = a(a+1)(a+2) \dots (a+r-1) = \Gamma(a+r) / \Gamma(a), \quad (a, 0) = 1 \quad (2.5)$$

and

$$((a), r) = (a_1, r)(a_2, r) \dots (a_A, r), \quad (2.6)$$

then, after some reduction, it is found that

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{((a), n) y^n}{((h), n) n!} [Q_{y+n}(x)]^2 \\ &= \pi [\Gamma(1+\nu) / \Gamma(3/2+\nu)]^2 [4(x^2-1)]^{-\nu-1} \\ &\times \sum_{n=0}^{\infty} \frac{(1+\nu, n)(1+\nu, n)(1+\nu, n)}{(2+2\nu, n)(3/2+\nu, n) n!} 1-x^2)^{-n} {}_{A+1}F_{H+1} \left(\begin{matrix} (a), -n \\ (h), 2+2\nu+n \end{matrix} ; -y \right). \end{aligned} \quad (2.7)$$

Useful results follow from this formula whenever the generalised hypergeometric function on the right of (2.7) is summable as a gamma product. All the summation formulae used below are to be found in Slater [3, p. 243, Appendix III].

3. SPECIAL CASES

Certain representative instances are now worked out in some detail, and the remaining formulae given are presented without proof. Let the inner series on the right of (2.7) take the form

$${}_2E_1(a, -n; 2+2\nu+n; -y), \quad (3.1)$$

If we let $y = -1$, then Vandermonde's theorem may be applied to the function (3.1) which becomes

$$\frac{(2+2\nu-a+n, n)}{(2+2\nu+n, n)} = \frac{(2+2\nu-a, 2n)(2+2\nu, n)}{(2+2\nu-a, n)(2+2\nu, 2n)}. \quad (3.2)$$

The general result (2.7) may then be written

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{(a, n)(-1)^n}{n!} [Q_{\nu+n}(x)]^2 \\ &= \pi [\Gamma(1+\nu)/\Gamma(3/2+\nu)]^4 [4(x^2-1)]^{-\nu-1} \\ & \times {}_4F_3 \left(\begin{matrix} 1+\nu, 1+\nu, 1+\nu-a/2, 3/2+\nu-a/2; \\ 3/2+\nu, 3/2+\nu, 2+2\nu-a \end{matrix} ; \frac{1}{1-x^2} \right). \end{aligned} \quad (3.3)$$

The series (3.1) may also be summed by Kummer's theorem if we put $y=1$ and $a=1+2\nu$, when

$${}_2F_1(1+2\nu, -n; 2+2\nu+n; -1) = (2+2\nu, n) / (3/2+\nu, n). \quad (3.4)$$

On substituting this expression into (2.7), we see that

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{(1+2\nu, n)}{n!} [Q_{\nu+n}(x)]^2 \\ &= \pi [\Gamma(1+\nu)/\Gamma(3/2+\nu)]^2 [4(x^2-1)]^{-\nu-1} \\ & \times {}_3F_2(1+\nu, 1+\nu, 1+\nu; 3/2+\nu, 3/2+\nu; 1/(1-x^2)). \end{aligned} \quad (3.5)$$

The formula

$$\begin{aligned} & {}_5F_4 \left(\begin{matrix} a, 1+a/2, b, c, -n; \\ a/2, 1+a-b, 1+a-c, 1+a+n \end{matrix} ; 1 \right) \\ &= \frac{(1+a, n)(1+a-b-c, n)}{(1+a-b, n)(1+a-c, n)}, \end{aligned} \quad (3.6)$$

(Slater [3. p. 244]) is one of the more complicated results which may be applied to sum the hypergeometric function on the right of

(2.7) . Put $y = -1$, $a_1 = 1 + 2v$, $a_2 = 3/2 + v$, $a_3 = b$, $a_4 = c$, $h_1 = 1/2 + v$, $h_2 = 2 + 2v - b$ and $h_3 = 2 + 2v - c$, when the hypergeometric function under consideration becomes equal to

$$\frac{(2+2v, n) (2+2v-b-c, n)}{(2+2v-b, n) (2+2v-c, n)} \quad (3.7)$$

and we obtain the formula

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{(1+2v, n) (3/2+v, n) (b, n) (c, n) (-1)^n}{(1/2+v, n) (2+2v-b, n) (2+2v-c, n) n!} [Q_{v+n}(x)]^2 \\ &= \pi [\Gamma(1+v) / \Gamma(3/2+v)]^2 [4(x^2-1)]^{-v-1} \\ & \times {}_4F_3 \left(\frac{1+v, 1+v, 1+v, 2+2v-b-c}{3/2+v, 2+2v-b, 2+2v-c}; 1/(1-x^2) \right). \end{aligned} \quad (3.8)$$

As an example of a further special case, if we put $c = 2 + 2v - b$, the ${}_4F_3$ series on the right reduces to unity, giving the rather curious result :

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(1+2v, n) (3/2+v, n) (-1)^n}{(1/2+v, n) n!} [Q_{v+n}(x)]^2 &= \pi [\Gamma(1+v) / \Gamma(3/2+v)]^2 \\ & [4(x^2-1)]^{-v-1}. \end{aligned} \quad (3.9)$$

Other similar formulas may be obtained by using the summation theorems for the generalised hypergeometric series, which are to be found in Slater [3, Appendix III] . These new expressions are listed below and the corresponding summation theorem is indicated for each case .

$$\sum_{n=0}^{\infty} \frac{(1+2v, n) (b, n) (-1)^n}{(2+2v-b, n) n!} [Q_{v+n}(x)]^2$$

$$= \pi [\Gamma(1+\nu) / \Gamma(3/2+\nu)]^2 [4(x^2-1)]^{-\nu-1} \quad (\text{III.9})$$

$$\times {}_4F_3 \left(\begin{matrix} 1+\nu, 1+\nu, 1+\nu, 3/2+\nu-b; \\ 3/2+\nu, 3/2+\nu, 2+2\nu-b; \end{matrix} ; \frac{1}{1-x^2} \right). \quad (3.10)$$

$$\sum_{n=0}^{\infty} \frac{(1+2\nu, n) (3/2+\nu, n)}{(1/2+\nu, n) n!} [Q_{\nu+n}(x)]^2 \quad (\text{III.25})$$

$$= \pi [\Gamma(1+\nu) / \Gamma(3/2+\nu)]^2 [4(x^2-1)]^{-\nu-1}$$

$$\times {}_2F_1 \left(\begin{matrix} 1+\nu, 1+\nu; \\ 3/2+\nu; \end{matrix} ; \frac{1}{1-x^2} \right), \quad (3.11)$$

$$\sum_{n=0}^{\infty} \frac{(1+2\nu, n) (3/2+\nu, n) (b, n)}{(1/2+\nu, n) (2+2\nu-b, n) n!} [Q_{\nu+n}(x)]^2$$

$$= \pi [\Gamma(1+\nu) / \Gamma(3/2+\nu)]^2 [4(x^2-1)]^{-\nu-1}. \quad (\text{III.11})$$

$$\times {}_3F_2 (1+\nu, 1+\nu, 1+\nu; 3/2+\nu, 2+2\nu-b; 1/(1-x^2)) \quad (3.12)$$

and

$$\sum_{n=0}^{\infty} \frac{(1+2\nu, n) (3/2+\nu, n) (b, n) (-1)^n}{(1/2+\nu, n) (2+2\nu-b, n) n!} [Q_{\nu+n}(x)]^2$$

$$= \pi [\Gamma(1+\nu) / \Gamma(3/2+\nu)]^2 [4(x^2-1)]^{-\nu-1} \quad (\text{III.26})$$

$$\times {}_4F_3 \left(\begin{matrix} 1+\nu, 1+\nu, 1+\nu, 3/2+\nu-b; \\ 3/2+\nu, 2+2\nu-b, 2+2\nu-b; \end{matrix} ; \frac{1}{1-x^2} \right). \quad (3.13)$$

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APPLICATIONS OF FIXED POINT THEOREMS

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Several interesting results using fixed point theory are given in Approximation Theory. An excellent reference is Cheney [3]. For a survey paper one is referred to [9]

The following theorem due to Brosowski [1] has been a basic important result .

Let f be a contractive linear operator of a normed linear space X . Let C be an f -invariant subset of X and x an f -invariant point. . If the set of best C approximants to x is nonempty, compact and convex, then it contains an f -invariant point .

In [10] Singh observed that the linearity of f and the convexity of the set of best C -approximants to x , can be relaxed .

Subrahmanyam [12], S. P. Singh [10], [11] : and K. L. Singh [9] deal with fixed point theorems of nonexpansive mappings. Carbone [2] and Rao and Mariados [7] deal with generalized non-expansive mappings, Sahab, Khan and Sessa [8] extended the result of [10] in a different direction whereas Hicks and Humphries relaxed the boundary condition in their paper [5] .

The purpose of this paper is to give further results unifying earlier known results.

We need the following

Let $f: X \rightarrow X$. Then f is called nonexpansive mapping if

$$\|fx - fy\| \leq \|x - y\| \text{ for all } x, y \in X.$$

If $\|fx - p\| \leq \|x - p\|$ for all $x \in X$ and $p \in F(f)$, then f is called a quasi nonexpansive map ($F(f)$ stands for the set of fixed points of f).

If $\|fx - fy\| \leq \alpha \|x - y\| + \beta (\|x - fx\| + \|y - fy\|) + \gamma (\|x - fy\| + \|y - fx\|)$ for all $x, y \in X$, $\alpha, \beta, \gamma \geq 0$ and $\alpha + 2\beta + 2\gamma \leq 1$, then f is called a generalized nonexpansive mapping.

A Banach space X is said to satisfy Opial's condition if the following holds. If a sequence $\{x_n\}$ converges weakly to $x_0 \in X$ and if $x \neq x_0$ then

$$\liminf_{n \rightarrow \infty} \|x_n - x\| > \liminf_{n \rightarrow \infty} \|x_n - x_0\|.$$

Hilbert spaces and l^p ($1 \leq p < \infty$) spaces satisfy Opial's condition.

A mapping $I-f$ is semiclosed if $x_n \rightarrow x$ weakly and $(I-f)x_n$ converges to y strongly imply that $y = (I-f)x$.

A subset C of a normed linear space X is said to be strashaped with respect to a point $p \in C$ if for all $x \in C$ and all $\lambda, 0 \leq \lambda \leq 1$, $\lambda x + (1-\lambda)p \in C$.

If X is a normed linear space, B is a subset of X and $x \in X$, then

$$d(x, B) = \inf \{ \|x - y\| : y \in B \}$$

denotes the distance of x to the set B .

The set of all best approximations to x is denoted by

$$D(x, B) = \{y \in B : \|x - y\| = d(x, B)\} \quad [3].$$

Now we give our results.

Theorem 1. Let C be a subset of a Banach space X and $T : X \rightarrow X$ be a mapping. . Let $T(\partial C) \subset C$ (∂C stands for the boundary of C) . Let $Tx_0 = x_0$, $x_0 \in X$ and T satisfy the following condition :
 $\|Tx - Ty\| \leq \alpha \|x - y\| + \beta (\|x - Tx\| + \|y - Ty\|) + \gamma (\|x - Ty\| + \|y - Tx\|)$
 for all $x, y \in X$, where α, β and γ are reals with $\alpha + 2\beta + 2\gamma \leq 1$.

Further, if the set D , the set of best C -approximants to x_0 , is nonempty, closed, convex $T(D)$ compact and T is continuous on D then T has a fixed point in D closest to x_0 .

Remarks. In [1] a similar result is proved for a linear, nonexpansive map T from C to itself, and an extended version in [10] where $T : C \rightarrow C$ is taken, Here we do not require linearity, nonexpansiveness of T on X and $T : C \rightarrow C$ not needed only $T(\partial C) \subset C$ serves the purpose. We relax compactness of C .

The following theorem (A) will be used in our proof.

Theorem A. If $f : C \rightarrow C$ is nonexpansive, where C is a closed, convex subset of a normed linear space X and $\overline{f(C)}$ is compact, then f has a fixed point [4].

Note : Theorem A holds in C is starshaped .

Proof. First we show that

$$\|Tx - Tx_0\| \leq \|x - x_0\| \quad \text{for all } x \in X.$$

Clearly,

$$\begin{aligned} \|Tx - Tx_0\| &\leq \alpha \|x - x_0\| + \beta (\|x - Tx\| + \|x_0 - Tx_0\|) + \gamma (\|x - Tx_0\| \\ &\quad + \|x_0 - Tx\|) . \end{aligned}$$

Since $Tx_0 = x_0$, we get

$$\|Tx - Tx_0\| \leq \alpha \|x - x_0\| + \beta \|x - Tx\| + \gamma (\|x - x_0\| + \|Tx_0 - Tx\|) .$$

Now,

$$\|x - Tx\| \leq \|x - x_0\| + \|Tx - Tx_0\|.$$

therefore we have

$$\|Tx - Tx_0\| \leq \frac{\alpha + \beta + \gamma}{1 - \beta - \gamma} \|x - x_0\|,$$

$$i. e. \quad \|Tx - Tx_0\| \leq \|x - x_0\| \quad \text{for all } x \in X.$$

We claim that $T : D \rightarrow D$.

Let $y \in D$. Then

$$\|Ty - x_0\| = \|Ty - Tx_0\| \leq \|y - x_0\| \quad \text{by (1)}.$$

Since y is nearest to x_0 so Ty is also nearest to x_0 , i.e. $Ty \in D$.

Since D is a nonempty closed, convex, $\overline{T(D)}$ compact subset of X and $T : D \rightarrow D$ is a continuous map therefore T has a fixed point by Schauder Fixed Point Theorem. So $Tx = x$ say, where $x \in D$ closest to x_0 .

Corollary, In case we have a quasinonexpansive map in place of generalized nonexpansive map, the result still remains true.

Note. The result due to Carbone [2] follows as corollary. Here the condition of compactness is relaxed.

Theorem 2. Let $T : X \rightarrow X$ be a mapping on a Banach space X satisfying the following :

- I) $Tx_0 = x_0, \quad x_0 \in X$.
- II) $T\{\partial C\} \subset C$, where C is a subset of X .
- III) The set D : the set of best C -approximants to x_0 is closed, starshaped, and $\overline{R(D)}$ is compact.
- IV) T is nonexpansive on D and $\|Tx - Tx_0\| \leq \|x - x_0\|$ for all $x \in D$.

Then T has a fixed point in D closest to x_0 .

Our proof follows similar lines as in [10].

Proof. As in Theorem 1 it is easy to show that $T : D \rightarrow D$.

Since T is a nonexpansive mapping on D and D is starshaped with star center P therefore we define

$$T_{k_n} x = k_n T x + (1 - k_n) p \text{ for all } x \in D.$$

where $\{k_n\}$ is a sequence of reals, $0 \leq k_n < 1$, converges to 1 as $n \rightarrow \infty$.

Now, $T_n : D \rightarrow D$ for each n , since so is T (we write n in place of k_n).

Also, $\|T_n x - T_n y\| = k_n \|Tx - Ty\| \leq k_n \|x - y\|$, since T is nonexpansive on D . Each T_n is a contraction map with k_n contraction constant. Then each T_n has a unique fixed point say y_n , for $n = 1, 2, \dots$, i. e.

$$T_n y_n = y_n, \quad n = 1, 2, \dots$$

Since $\overline{T(D)}$ is compact therefore $\{y_n\}$ has a subsequence $\{y_{n_i}\}$ converging to y say.

We claim that $Ty = y$ in D .

Since

$$y_{n_i} = T_{n_i} y_{n_i} = (1 - k_{n_i}) p + k_{n_i} T y_{n_i},$$

taking limit as $i \rightarrow \infty$ and using the fact that T is continuous we get that

$$y = Ty \text{ since } k_{n_i} \rightarrow 1.$$

Theorem 3. Let X be a Banach space with Opial's condition and $T : X \rightarrow X$ be a map. Let $Tx_0 = x_0$, $x_0 \in X$, $T(\partial C) \subset C$ where C is a nonempty, weakly compact, starshaped subset of X and T is nonexpansive on $DU\{x_0\}$. Then T has a fixed point in D closest to x_0 .

Proof. It is easy to show that $T : D \rightarrow D$.

Now, define

$$f_n x = k_n f x + (1 - k_n) p.$$

In this case each f_n is contraction with constant $k_n < 1$, so we have $f_n x_n = x_n$ for each n by Banach contraction principle.

Since $f : D \rightarrow D$ so is $f_n : D \rightarrow D$. As D is weakly compact we get that $\{x_n\}$ has a subsequence $\{x_{n_i}\}$ converging weakly to y say $x_{n_i} \rightarrow y$.

$$x_{n_i} - f x_{n_i} = (1 - k_{n_i}) p + f x_{n_i} (1 - k_{n_i}),$$

so

$$x_{n_i} - f x_{n_i} \rightarrow 0 \text{ as } i \rightarrow \infty \text{ since } k_{n_i} \rightarrow 1.$$

Since X satisfies Opial's condition so $I-f$ is demiclosed (if $x_n \rightarrow y$ and $(I-f)x_n \rightarrow z$ then $y \in D$ and $(I-f)y = z$).

Thus $(I-f)y = 0$ and $fy = y$.

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ON THE INTEGRATION OF THE SYSTEM OF PARTIAL
DIFFERENTIAL EQUATIONS ASSOCIATED WITH THE
TRIPLE HYPERGEOMETRIC FUNCTION $F_D^{(3)}$

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ABSTRACT

Interest in the Lauricella function of the fourth kind in several variables, $F_D^{(n)}$ has recently centred upon its SL-symmetry (Miller [5]). The cases where $n=1$ and $n=2$ have been well explored, but for higher values of n , no explicit study has been carried out. Fundamental systems of solutions are obtained in this paper for all the singular points of the associated partial differential system. These results are expressed entirely in terms of the functions $F_D^{(3)}$, $C_3^{(1)}$ and $C_3^{(2)}$, the functions $D_3^{1,2}$ and $D_3^{2,1}$ not being involved. This is important from the point of view of Lie theory as applied to the differential system under consideration in that no hypergeometric functions with restricted parameters, such as $D_3^{1,2}$, are required.

1. INTRODUCTION

In the integration of the partial differential system associated with the Lauricella function $F_D^{(n)}$ by the use of Pochhammer integrals,

solutions in terms of this function and the functions $C_n^{(k)}$, all with $n+2$ parameters, are obtained. Furthermore, when $n \geq 3$, additional solutions present themselves, the so-called $D_n^{p,q}$ functions, which possess $n+3$ parameters with one restriction. It has hitherto tacitly assumed that all the above types of functions would be necessary in constructing fundamental sets of integrals relative to all the singular points of the partial differential system under consideration (Exton [1, p. 106]). It will be shown below that the functions with restricted parameters need not necessarily be included in such sets of integrals.

The importance of this arises from the treatment of the problem of integrating the system by means of Lie theory. The function $F_D^{(n)}$ possesses SL-symmetry which has aroused some interest of late (Miller [5]), and any solutions in the form of functions with restricted parameters, the $D_n^{p,q}$ functions in this case, present difficulties in the application of Lie theory in this context. (Hrabowski [2] and [3]). From the results of this study, this difficulty is removed when $n=3$, and it is conjectured that the same applies for higher values of n also.

The derivation is based upon a combination of the use of series manipulation (Olsson [6]) and upon the use of the general properties of certain of the solutions of the system of partial differential equations under consideration (Olsson [7]). This system was first formed by Lauricella [4] (see also Srivastava and Karlsson [10, p. 293 *et seq.*]):

$$\begin{aligned}
& x_1 (1-x_1) \frac{\partial^2 F}{\partial x_1^2} + x_2 (1-x_1) \frac{\partial^2 F}{\partial x_1 \partial x_2} + x_3 (1-x_1) \frac{\partial^2 F}{\partial x_1 \partial x_3} \\
& + [c - (a+b_1+1)x_1] \frac{\partial F}{\partial x_1} - b_1 x_2 \frac{\partial F}{\partial x_2} - b_1 x_3 \frac{\partial F}{\partial x_3} - ab_1 F = 0, \\
& x_2 (1-x_2) \frac{\partial^2 F}{\partial x_2^2} + x_1 (1-x_2) \frac{\partial^2 F}{\partial x_2 \partial x_1} + x_3 (1-x_2) \frac{\partial^2 F}{\partial x_2 \partial x_3} \\
& + [c - (a+b_2+1)x_2] \frac{\partial F}{\partial x_2} - b_2 x_1 \frac{\partial F}{\partial x_1} - b_2 x_3 \frac{\partial F}{\partial x_3} - ab_2 F = 0, \\
& x_3 (1-x_3) \frac{\partial^2 F}{\partial x_3^2} + x_1 (1-x_3) \frac{\partial^2 F}{\partial x_3 \partial x_1} + x_2 (1-x_3) \frac{\partial^2 F}{\partial x_3 \partial x_2} \\
& + [c - (a+b_3+1)x_3] \frac{\partial F}{\partial x_3} - b_3 x_1 \frac{\partial F}{\partial x_1} - b_3 x_2 \frac{\partial F}{\partial x_2} - ab_3 F = 0. \quad (1.1)
\end{aligned}$$

Lauricella [4] has also shown that the general integral of (1.1) may be presented as a linear combination of four of its independent integrals.

The functions $F_D^{(n)}$ and $C_n^{(k)}$ are given by

$$\begin{aligned}
& F_D^{(n)}(a, b_1, b_2, \dots, b_n; c; x_1, x_2, \dots, x_n) \\
& = \sum_{m_1, \dots, m_n=0}^{\infty} \frac{(a, m_1 + \dots + m_n) (b_1, m_1) \dots (b_n, m_n)}{(c, m_1 + \dots + m_n) m_1! \dots m_n!} x_1^{m_1} \dots x_n^{m_n}, \\
& \quad (1.2)
\end{aligned}$$

(Lauricella [4]),

$$C_n^{(k)}(b_1, b_2, \dots, b_n, n, a'; x_1, x_2, \dots, x_n)$$

$$= \sum_{m_1, \dots, m_n=0}^{\infty} (b_1, m_1) \dots (b_n, m_n) (a, m_{k+1}, \dots, m_n - m_1 - \dots - m_k) \\ (a', m_1 + \dots + m_k - m_{k+1} - \dots - m_n) \frac{x_1^{m_1}}{m_1!} \dots \frac{x_n^{m_n}}{m_n!} \quad (1.3)$$

(Exton [1, p. 104]).

Both series converge if $|x_j| < 1, j=1, 2, \dots, n$,

The Pochhammer symbol (a, n) is given by

$$(a, n) = a(a+1) \dots (a+n-1) = \Gamma(a+n) / \Gamma(a); (a, 0) = 1.$$

The singular manifolds of the partial differential system (1.1) are easily seen to be

$$x_i = 0, 1, \infty \text{ and } x_j = x_i, i, j = 1, 2, \dots, n, i \neq j.$$

The singularities are formed by the intersection of two or more of the singular manifolds.

2. THE USE OF SERIES MANIPULATION

The triple series representation of $F_D^{(3)}$ may be written as a double series of Gauss functions :

$$F_D^{(3)}(a, b_1, b_2, b_3; c; x_1, x_2, x_3) \\ = \sum_{m_1, m_2=0}^{\infty} \frac{(a, m_1+m_2)(b_1, m_1)(b_2, m_2)}{(c, m_1+m_2)m_1!m_2!} x_1^{m_1} x_2^{m_2} {}_2F_1\left(\begin{matrix} a+m_1+m_2, b_3 \\ c+m_1+m_2 \end{matrix}; x_3\right). \quad (3.1)$$

The analytic continuation of the ${}_2F_1$ series near the point at infinity is

$${}_2F_1(a, b; c; x) = \frac{\Gamma(c) \Gamma(b-a)}{\Gamma(b) \Gamma(c-a)} (-x)^{-a} {}_2E_1(a, a-c+1; a-b+1; 1/x) \\
 + \frac{\Gamma(c) \Gamma(a-b)}{\Gamma(a) \Gamma(c-b)} (-x)^{-b} {}_2F_1(b, b-c+1; b-a+1; 1/x). \quad (2.2)$$

$$| \arg(-x) | < \pi.$$

(See Whittaker and Watson [11, p. 289]).

The terms on the right of (2.2) may be inserted into (2.1) in order to construct new solutions of (1.1). Omitting the algebra, we see that the first term yields the solution

$$x_3^{-a} F_D^{(3)}(a, b_1, b_2, a-c+1; a-b_3+1; x_1/x_3, x_2/x_3, 1/x_3). \quad (2.3)$$

If the above process is repeated, but using the second term on the right of (2.2), we obtain the solution

$$x_3^{1-c} C_3^{(2)}(b_1, b_2, a-c+1, b_3-c+1, c-1; -x_1/x_3, -x_2/x_3, -x_3). \quad (2.4)$$

Further application of similar technique gives the additional solutions

$$x_3^{-b_3} C_3^{(2)}(b_1, b_2, b_3, b_3-c+1, a-b_3; x_1/x_3, x_2/x_3, -1/x_3), \quad (2.5)$$

$$x_2^{-b_2} x_3^{-b_3} C_3^{(1)}(b_1, b_2, b_3, 1+b_2+b_3-c, a-b_2-b_3; -x_1, -1/x_2, -1/x_3), \quad (2.6)$$

$$x_2^{b_3-a} x_3^{-b_3} C_3^{(2)}(b_1, 1+a-c, b_3, b_2+b_3-a, a-b_3; -x_1/x_2, -1/x_2, -x_2/x_3), \quad (2.7)$$

$$x_1^{-b_1} x_2^{-b_2} x_3^{-b_3} F_D^{(3)}(b_1+b_2+b_3-c+1, b_1, b_2, b_3; b_1+b_2+b_3-a+1; 1/x_1, 1/x_2, 1/x_3) \quad (2.8)$$

$$\text{and } F_D^{(3)}(a, b_1, b_2, b_3; 1+a+b_1+b_2+b_3-c; 1-x_1, 1-x_2, 1-x_3) \quad (2.9)$$

Other solutions might be obtained in like manner, but it is more convenient in the present context to make use of the general properties of the solutions of the partial differential system (1.1).

3 GENERAL PROPERTIES OF THE SYSTEM

ASSOCIATED WITH $F_D^{(3)}$

It will be seen by inspection of the right-hand member of (1.2) that the function $F_D^{(3)}$ is unchanged if, for example, (x_1, b_1) and (x_3, b_3) are interchanged. Hence it follows that changes of this type may be freely applied to the system (1.1), so that, if $Z(a, b_1, b_2, b_3; c; x_1, x_2, x_3)$ is a solution of (2.1), then so also are $Z(a, b_3, b_2, b_1; c; x_3, x_2, x_1)$ etc. Similarly, from the fact that (2.3), (2.8) and (2.9) are solutions of (1.1) of the form $F_D^{(3)}$ we may say that $x_3^{-a} Z(a, b_1, b_2, a-c+1; a-b_3+1; x_1/x_3, x_2/x_3, 1/x_3)$, $x_1^{-b_1} x_2^{-b_2} x_3^{-b_3} Z(b_1+b_2+b_3-c+1, b_1, b_2, b_3; b_1+b_2+b_3-a+1; 1/x_1, 1/x_2, 1/x_3)$ and $Z(a, b_1, b_2, b_3; 1+a+b_1+b_2+b_3-c; 1-x_1, 1-x_2, 1-x_3)$ are also solutions of the system (1.1). Whether or not any solutions deduced in this fashion are independent from any others must be checked separately in each case.

This technique has been employed very effectively by Olsson [7] in his investigation of the partial differential system associated with the Appell function F_2 . By way of illustration, we obtain the solution

$$x_1^{-b_1} x_2^{-b_2} x_3^{b_1+b_2-c+1} F_D^{(3)} (b_1+b_2+b_3-c+1, b_1, b_2, a-c+1; 2+b_1+b_2-c; x_3/x_1, x_3/x_2, x_3) \quad (3.1)$$

by the application of (2.8) to (2.3). Another type of general solution may now be written down :

$$x_1^{-b_1} x_2^{-b_2} x_3^{b_1+b_2-c+1} Z(b_1+b_2+b_3-c+1, b_1, b_2, a-c+1; 2+b_1+b_2-c; x_3/x_1, x_3/x_2, x_3) . \quad (3.2)$$

4. FUNDAMENTAL SYSTEMS OF SOLUTIONS OF (1.1)

By employing the results given above, it is possible to write down fundamental sets of integrals of (1.1) valid near all of the singular points of the system. It might be observed that the only singularities arising from the intersection of three singular manifolds are of the type $(\infty, 1, 0)$ and, as such, a complete fundamental system might be constructed, each member of which is a series convergent in the whole neighbourhood of the point in question. Otherwise, such points as $(0, 1, 1)$ and $(0, 0, 0)$ for example, are formed from the intersection of four and six singular manifolds respectively. Hence, we would expect that some of the members of each fundamental system would possess series representation which are convergent in only a part of the neighbourhood concerned. The remaining parts of each neighbourhood may be dealt with by interchanging the subscripts as indicated in Section 3. We now record the following fundamental systems, each valid near the singularity indicated with any restrictions as are appropriate (see also Srivastava [8] and [9]):

(0,0,0)

$$F_D^{(3)} (a, b_1, b_2, b_3, c; x_1, x_2, x_3) ,$$

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$$x_1^{-b_1} x_2^{-b_2} x_3^{b_1+b_2-c+1} F_D^{(3)} (b_1+b_2+b_3-c+1, b_1, b_2, a-c+1; b_1+b_2-c; x_3/x_1, x_3/x_2, x_2),$$

$$x_1^{-b_1} x_2^{-b_2} C_3^{(2)} (b_1, b_2, b_3, a-b_1-b_2, b_1+b_2-c+1; x_3/x_1, x_3/x_2, -x_3),$$

$$x_1^{-b_1} x_2^{b_1-c+1} C_3^{(2)} (b_1, 1+a-c, b_3, c-b_1-1, b_1+b_2-c+1; x_2/x_1, -x_2, -x_3/x_2).$$

$$|x_1| > |x_2|, |x_2| > |x_3|, |x_1| > |x_3|.$$

($\infty, 0, 0$)

$$x_1^{-b_1} x_2^{-b_2} x_3^{b_1+b_2-c+1} F_D^{(3)} (b_1+b_2+b_3-c+1, b_1, b_2, a-c+1; b_1+b_2-c; x_3/x_1, x_3/x_2, x_3),$$

$$x_1^{-b_1} x_2^{-b_2} C_3^{(2)} (b_1, b_2, b_3, a-b_1-b_2, b_1+b_2-c+1; x_3/x_1, x_3/x_2, -x_3),$$

$$x_1^{-b_1} C_3^{(1)} (b_1, b_2, b_3, a-b_1, 1+b_1-c; -1/x_1, -x_2, -x_3),$$

$$x_1^{-b_1} x_2^{b_1-c+1} C_3^{(1)} (b_1, 1+a-c, b_3, c-b_1-1, b_1+b_2-c+1; -x_2/x_1, -x_2, -x_3/x_2).$$

$$|x_2| > |x_3|.$$

($0, \infty, \infty$)

$$x_1^{-a} F_D^{(3)} (a, b_1, b_2, a-c+1; a-b_3+1; x_1/x_3, x_2/x_3, 1/x_3),$$

$$x_3^{-b_3} C_3^{(2)} (b_1, b_2, b_3, b_3 - c + 1, a - b_3; x_1/x_3, x_2/x_3, -1/x_3),$$

$$x_2^{-b_2} x_3^{-b_3} C_3^{(1)} (b_1, b_2, b_3, 1 + b_2 + b_3 - c, a - b_2 - b_3; -x_1, -1/x_2, -1/x_3),$$

$$x_2^{b_3 - a} x_3^{-b_3} C_3^{(2)} (b_1, 1 + a - c, b_3, b_2 + b_3 - a, a - b_3; -x_1/x_2, -1/x_2, -x_2/x_3),$$

$$(\infty, \infty, \infty)$$

$$x_3^{-a} F_D^{(3)} (a, b_1, b_2, a - c + 1; a - b_3 + 1; x_1/x_3, x_2/x_3, 1/x_3),$$

$$x_1^{-b_1} x_2^{-b_2} x_3^{-b_3} F_D^{(3)} (b_1 + b_2 + b_3 - c + 1, b_1, b_2, b_3; b_1 + b_2 + b_3 - a + 1; \\ 1/x_1, 1/x_2, 1/x_3),$$

$$x_3^{-b_3} C_3^{(2)} (b_1, b_2, b_3, b_3 - c + 1, a - b_3; x_1/x_3, x_2/x_3, -1/x_3),$$

$$x_2^{b_3 - a} x_3^{-b_3} C_3^{(2)} (b_1, 1 + a - c, b_3, b_2 + b_3 - a, a - b_3;$$

$$x_1/x_2, -1/x_2, -x_2/x_3).$$

$$|x_3| > |x_1|, |x_3| > |x_2|, |x_2| > |x_1|.$$

$$(\infty, 1, 0)$$

$$x_1^{-b_1} x_2^{-b_2} x_3^{b_1 + b_2 - c + 1} F_D^{(3)} (b_1 + b_2 + b_3 - c + 1, b_1, b_2, a - c + 1;$$

$$b_1 + b_2 - c; x_3/x_1, x_3/x_2, x_3),$$

$$x_1^{-a} F_D^{(3)}(a, b_3, b_2, a-c+1; a-b_1+1; x_3/x_1, x_2/x_1, 1/x_2) .$$

$$x_1^{-b_1} x_2^{-b_2} C_3^{(2)}(b_1, b_2, b_3, a-b_1-b_2, b_1+b_2-c+1; x_3/x_1, x_3/x_2, x_3),$$

$$x_3^{-b_1} C_3^{(2)}(b_3, b_2, b_1, b_1-c+1, a-b_1; x_3/x_1, x_2/x_1, -1/x_1) .$$

$$(\infty, 1, 1)$$

$$(1-x_1)^{-b_1} (1-x_2)^{-b_2} (1-x_3)^{-b_3} F_D^{(3)}(c-a, b_1, a_2, c-b_1-b_2-b_3; c-a-b_3-1;$$

$$(1-x_3)/(1-x_1), (1-x_3)/(1-x_2), 1-x_3),$$

$$(1-x_1)^{-b_1} (1-x_2)^{-b_2} C_3^{(2)}(b_1, b_2, b_3, a-b_1-b_2, c-a; (1-x_3)/(1-x_1),$$

$$(1-x_3)/(1-x_2), x_3-1),$$

$$(1-x_1)^{-b_1} C_3^{(1)}(b_1, b_2, b_3, a-b_1, c-a-b_2-b_3; 1/(x_1-1), x_2-1, x_3-1),$$

$$(1-x_1)^{-b_1} (1-x_2)^{c-a-b_2-b_3-1} C_3^{(1)}(b_1, c-b_1-b_2-b_3, b_3, a+b_2+b_3-c, c-a-b_2;$$

$$(x_2-1)/(1-x_1), x_2-1, (x_3-1)/(1-x_2)) .$$

$$|1-x_2| > |1-x_3| .$$

$$(1, \infty, \infty)$$

$$(1-x_3)^{-a} F_D^{(3)}(a, b_1, b_2, c-b_1-b_2-b_3; a-b_3+1; (1-x_1)/(1-x_3),$$

$$(1-x_2)/(1-x_3), 1/(1-x_3)),$$

$$(1-x)^{-b_3} C_3^{(2)}(b_1, b_2, b_3, c-a-b_1-b_2, a-b_3; (1-x_1)/(1-x_3),$$

$$(1-x_2)/(1-x_3), -1/(1-x_3)),$$

$$(1-x_2)^{-b_2} (1-x_3)^{-b_3} C_3^{(1)} (b_1, b_2, b_3, c-a-b_3, a-b_2-b_3; 1-x_1, \\ -1/(1-x_2), -1/(1-x_3)),$$

$$(1-x_2)^{-b_2} (1-x_3)^{-b_3} C_3^{(2)} (b_1, c-b_1-b_2-b_3, b_3, b_2+b_3-a, a-b_3; \\ -(1-x_1)/(1-x_2), -1/(1-x_2), -(1-x_2)/(1-x_3)).$$

$$|1-x_2| > |1-x_3|.$$

(0,0,1)

$$(1-x_1)^{-b_1} (1-x_2)^{-b_2} (1-x_3)^{c-a-b_3} F_D^{(3)}(c-a, b_1, b_2, c-b_1-b_2-b_3; \\ c-a-b_3-1; (1-x_3)/(1-x_1), (1-x_3)/(1-x_2), 1-x_3),$$

$$(1-x_1)^{-b_1} (1-x_2)^{-b_2} C_3^{(2)} (b_1, b_2, b_3, a-b_1-b_2, c-a; (1-x_3)/(1-x_1), \\ (1-x_3)/(1-x_2), x_3-1),$$

$$x_3^{-b_3} x_2^{-b_2} x_1^{b_2+b_3-c+1} F_D^{(3)}(b_1+b_2+b_3-c+1, b_3, b_2, a-c+1; \\ b_2+b_3-c; x_1/x_3, x_1/x_2, x_1),$$

$$x_2^{-b_2} x_3^{-b_3} C_3^{(2)} (b_3, b_2, b_1, a-b_2-b_3, b_2+b_3-c+1; x_1/x_3, x_1/x_2, -x_1).$$

$$|x_2| > |x_1|.$$

(1,1,0)

$$x_1^{-b_1} x_2^{-b_2} x_3^{b_1+b_2-c+1} F_D^{(3)}(1+b_1+b_2+b_3-c, b_1, b_2, 1+a-c; \\ b_1+b_2; x_3/x_1, x_3/x_2, x_3),$$

$$x_1^{-b_1} x_2^{-b_2} C_3^{(2)} (b_1, b_2, b_3, a-b_1-b_2, 1+b_1+b_2+b_3-c; x_3/x_1, x_3/x_2, -x_3),$$

$$(1-x_1)^{c-a-b_1} (1-x_2)^{-b_2} (1-x_3)^{-b_3} F_D^{(3)}(c-a, b_3, b_2, c-b_1-b_2-b_3;$$

$$c-a-b_1-1; (1-x_1)/(1-x_3), (1-x_1)/(1-x_2), 1-x_1),$$

$$(1-x_2)^{-b_2} (1-x_3)^{-b_3} C_3^{(2)}(b_3, b_2, b_1, a-b_2, -b_3, c-a-b_1;$$

$$(1-x_1)/(1-x_3), (1-x_1)/(1-x_2), x_1-1).$$

$$|1-x_2| > |1-x_3|.$$

$$(1, 1, 1)$$

$$F_D^{(3)}(a, b_1, b_2, b_3; 1+b_1+b_2+b_3+a-c; 1-x_1, 1-x_2, 1-x_3),$$

$$(1-x_1)^{-b_1} (1-x_2)^{-b_2} (1-x_3)^{c-a-b_3} F_D^{(3)}(c-a, b_1, b_2, c-b_1-b_2-b_3;$$

$$c-a-b_3-1; (1-x_3)/(1-x_1), (1-x_3)/(1-x_2), 1-x_3),$$

$$(1-x_1)^{-b_1} (1-x_2)^{-b_2} C_3^{(2)}(b_1, b_2, b_3, a-b_1-b_2, c-a-b_3; (1-x_3)/(1-x_1),$$

$$(1-x_3)/(1-x_2), x_3-1).$$

$$(1-x_1)^{-b_1} (1-x_2)^{c-a-b_2-b_3} C_3^{(2)}(b_1, c-b_1-b_2-b_3, b_3, a+b_2+b_3-c, c-a-b_3;$$

$$-(1-x_2)/(1-x_1), x_2-1, -(1-x_3-1)/(1-x_2)).$$

$$|1-x_1| > |1-x_3| \quad |1-x_2| > |1-x_3| \quad |1-x_1| > |1-x_2|$$

The remaining singularities of the system may be dealt with by interchanging the subscripts in the previous results as appropriate. Fundamental sets of solutions involving convergent power series involving powers of $x_i - x_{j1}$, $i, j = 1, 2, 3, i \neq j$, may be deduced by the

use of the following Euler transformations of $F_D^{(3)}$ or any of their combinations using the arguments of Section 3.

$$\begin{aligned}
 & F_D^{(3)} (a, b_1, b_2, b_3; c; x_1, x_2, x_3) \\
 &= (1-x_1)^{-b_1} (1-x_2)^{-b_2} (1-x_3)^{-b_3} F_D^{(3)} (c-a, b_1, b_2, b_3; c; \\
 &\quad x_1 / (x_1-1), x_2 / (x_2-1), x_3 / (x_3-1)) \\
 &= (1-x_1)^{-a} F_D^{(3)} (a, c-b_1-b_2-b_3, b_2, b_3; c; x_1 / (x_1-1), (x_1-x_2) / (x_1-1), \\
 &\quad (x_1-x_3) / (x_1-1)) \\
 &= (1-x_2)^{-a} F_D^{(3)} (a, b_1, c-b_1-b_2-b_3, b_3; c; (x_2-x_1) / (x_2-1), x_2 / (x_2-1), \\
 &\quad (x_2-x_3) / (x_2-1)) \\
 &= (1-x_3)^{-a} F_D^{(2)} (a, b_1, b_2, c-b_1-b_2-b_3; c; (x_3-x_1) / (x_3-1), (x_3-x_2) / (x_3-1), \\
 &\quad x_3 / (x_3-1)). \quad (4.1)
 \end{aligned}$$

(See Lauricella [4]). Series involving $x_i - x_j$ may be useful in covering those cases where $x_i \sim x_j$ which might otherwise be intractable. as for example, the series

$$x_1^{-a} F_D^{(2)} (a, b_1, b_2, a-c+1; a-b_3+1; x_1 / x_3, x_2 / x_3, 1/x_3) \quad (4.2)$$

when $x_2 \sim x_3$. (See Olsson [6]).

It will be seen that complete fundamental systems of integrals applicable to all the singular points of the system of partial differential equations associated with $F_L^{(3)}$ may be constructed entirely from

the functions $F_D^{(3)}$, $C_3^{(1)}$ and $C_3^{(2)}$ without having recourse to the functions with restricted parameters $D_3^{1,2}$ and $D_3^{2,1}$. It is conjectured that the analogous situation applies in the case of the system $F_D^{(n)}$ for all values of n .

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KARLSSON'S MULTIPLE HYPERGEOMETRIC FUNCTION AND ITS CONFLUENT FORMS

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ABSTRACT

In the present paper, We derive confluent forms of Karlsson's multiple hypergeometric function and obtain their generating relations and integral representations with their applications in obtaining their recurrence relations.

1. INTRODUCTION

Recently, Chandel and Gupta [1] introduced intermediate Lauricella's multiple hypergeometric functions ${}^{(k)}F_{AC}^{(n)}$, ${}^{(k)}F_{AD}^{(n)}$, ${}^{(k)}F_{BD}^{(n)}$ and their confluent forms. Subsequently, Karlsson [4] also introduced one more possible intermediate Lauricella function ${}^{(h)}F_{CD}^{(n)}$ defined by

$$(1.1) \quad {}^{(k)}F_{CD}^{(n)} [a, b, b_1, \dots, b_k; c, c_{k+1}, \dots, c_n; x_1, \dots, x_n].$$

$$= \sum_{m_1, \dots, m_n=0}^{\infty} \frac{(a)_{m_1+\dots+m_n} (b)_{m_{k+1}+\dots+m_n} (b_1)_{m_1} \dots (b_k)_{m_k}}{(c)_{m_1+\dots+m_k} (c_{k+1})_{m_{k+1}} \dots (c_n)_{m_n}}$$

$$\frac{x_1^{m_1}}{m_1!} \dots \frac{x_n^{m_n}}{m_n!},$$

which for $k=0$ reduces to Lauricella's function $F_C^{(n)}$ and for $k=n$

reduces to Lauricella's function $F_D^{(n)}$ [5].

The purpose of this paper is to study the confluent hypergeometric functions of the fourth intermediate Lauricella's function of Karlsson [4]. We shall discuss special cases, generating relations, and integral representations with their applications in obtaining recurrence relations of these multiple hypergeometric functions.

2. CONFLUENT FORMS

For confluent forms, we consider

$$(2.1) \quad \lim_{b_1, \dots, b_k \rightarrow \infty} {}^{(k)}F_{CD}^{(n)} [a, b, b_1, \dots, b_k; c, c_{k+1}, \dots, c_n;$$

$$\frac{x_1}{b_1}, \dots, \frac{x_k}{b_k}, x_{k+1}, \dots, x_n]$$

$$= \sum_{m_1, \dots, m_n=0}^{\infty} \frac{{}^{(a)}_{m_1+\dots+m_n} {}^{(b)}_{m_{k+1}+\dots+m_n}}{({}^{(c)}_{m_1+\dots+m_k} ({}^{(c_{k+1})}_{m_{k+1}} \dots ({}^{(c_n)}_{m_n})}$$

$$\frac{x_1^{m_1}}{m_1!} \dots \frac{x_n^{m_n}}{m_n!}$$

$$= {}^{(k)}\phi_{(1)CD}^{(n)} [a, b; c, c_{k+1}, \dots, c_n; x_1, \dots, x_n], k \neq 0.$$

For $k=0$, it reduces to Lauricella's $F_C^{(n)}$.

$$(2.2) \quad \lim_{b \rightarrow \infty} {}^{(k)}F_{CD}^{(n)} [a, b, b_1, \dots, b_k; c, c_{k+1}, \dots, c_n; x_1, \dots, x_k,$$

$$\frac{x_{k+1}}{b}, \dots, \frac{x_n}{b} \Bigg]$$

$$= \sum_{m_1, \dots, m_n=0}^{\infty} \frac{(a)_{m_1+\dots+m_n} (b_1)_{m_1} \dots (b_k)_{m_k}}{(c)_{m_1+\dots+m_k} (c_{k+1})_{m_{k+1}} \dots (c_n)_{m_n}} \frac{x_1^{m_1}}{m_1!} \dots \frac{x_n^{m_n}}{m_n!}$$

$$= {}_{(2)}\phi_{(k)}^{(n)} [a, b_1, \dots, b_k; c, c_{k+1}, \dots, c_n; x_1, \dots, x_n], k \neq n.$$

For $k=n$, it reduces to Lauricella's $F_D^{(n)}$,

$$(2.3) \lim_{a \rightarrow \infty} {}_{(k)}F_{CD}^{(n)} [a, b, b_1, \dots, b_k; c, c_{k+1}, \dots, c_n; \frac{x_1}{a}, \dots, \frac{x_n}{a}]$$

$$= \sum_{m_1, \dots, m_n=0}^{\infty} \frac{(b)_{m_{k+1}+\dots+m_n} (b_1)_{m_1} \dots (b_k)_{m_k}}{(c)_{m_1+\dots+m_k} (c_{k+1})_{m_{k+1}} \dots (c_n)_{m_n}} \frac{x_1^{m_1}}{m_1!} \dots \frac{x_n^{m_n}}{m_n!}$$

$$= {}_{(2)}\phi_{(k)}^{(n)} [b, b_1, \dots, b_k; c, c_{k+1}, \dots, c_n; x_1, \dots, x_n], k \neq 0, k \neq n.$$

For $k=n$ it reduces to $\phi_{(2)}^{(n)}$ while for $k=0$ it reduces to $\Psi_2^{(n)}$.

$$(2.4) \lim_{c_{k+1}, \dots, c_n \rightarrow 0} {}_{(k)}F_{CD}^{(n)} [a, b, b_1, \dots, b_k; c, c_{k+1}, \dots, c_n;$$

$$x_1, \dots, x_k, c_{k+1} x_{k+1}, \dots, c_n x_n]$$

$$= \sum_{m_1, \dots, m_n=0}^{\infty} \frac{(a)_{m_1+\dots+m_n} (b)_{m_{k+1}+\dots+m_n} (b_1)_{m_1} \dots (b_k)_{m_k}}{(c)_{m_1+\dots+m_k}} \frac{x_1^{m_1}}{m_1!} \dots \frac{x_n^{m_n}}{m_n!}$$

$$= {}^{(k)}\phi_{CD}^{(n)} [a, b, b_1, \dots, b_k; c; x_1, \dots, x_n] \quad k \neq n.$$

For $k=n$, it reduces to Lauricella's $F_D^{(n)}$.

3. INTEGRAL REPRESENTATIONS

Making an appeal to Srivastava [6, p. 101]

$$(3.1) \quad (\lambda, m) = \frac{1}{\Gamma(\lambda)} \int_0^{\infty} e^{-t} t^{\lambda+m-1} dt,$$

and Erdélyi [2, p. 13]

$$(3.2) \quad \frac{1}{(\lambda, m)} = \frac{\Gamma(\lambda)}{2\pi i} \int_{-\infty}^{(0+)} e^t t^{-\lambda-m} dt,$$

$$\operatorname{Re}(\lambda) > 0, m=0, 1, 2, \dots,$$

we derive the following integral representations :

$$(3.3) \quad {}^{(k)}F_{CD}^{(n)} [a, b, b_1, \dots, b_k; c, c_{k+1}, \dots, c_n; x_1, \dots, x_n]$$

$$= \frac{1}{2\pi i} \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \int_0^{\infty} \int_0^{\infty} \int_{-\infty}^{(0+)} e^{-(s+t+u)} s^{a-1} t^{b-1} u^{-c} (1-sx_1/u)^{-b_1}$$

$$\dots (1-sx_k/u)^{-b_k} {}_0F_1 [-; c_{k+1}; stx_{k+1}] \dots {}_0F_1 [-; c_n; stx_n] ds dt du,$$

where $\operatorname{Re}(a) > 0, \operatorname{Re}(b) > 0$.

$$(3.4) \quad {}_{(1)}^{(k)}\phi_{CD}^{(n)} [a, b; c, c_{k+1}, \dots, c_n; x_1, \dots, x_n]$$

$$= \frac{1}{2\pi i} \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \int_0^\infty \int_0^\infty \int_{-\infty}^{(0+)} e^{-(s+t-u)} e^{s/u(x_1 + \dots + x_k)}$$

$$s^{a-1} t^{b-1} u^{-c}$$

$$\cdot {}_0F_1 [-; c_{k+1}; st x_{k+1}] \dots {}_0F_1 [-; c_n; st x_n] ds dt du,$$

where $\text{Re}(a) > 0$, $\text{Re}(b) > 0$,

$$(3.5) \quad {}_{(2)}^{(k)}\phi_{CD}^{(n)} [a, b_1, \dots, b_k; c, c_{k+1}, \dots, c_n; x_1, \dots, x_n]$$

$$= \frac{1}{2\pi i} \frac{\Gamma(c)}{\Gamma(a)} \int_0^\infty \int_{-\infty}^{(0+)} e^{-s-u} s^{a-1} u^{-c} (1-sx_1/u)^{-b_1} \dots (1-sx_k/u)^{-b_k}$$

$$\cdot {}_0F_1 [-; c_{k+1}; sx_{k+1}] \dots {}_0F_1 [-; c_n; sx_n] ds du,$$

where $\text{Re}(a) > 0$.

$$(3.6) \quad {}_{(3)}^{(k)}\phi_{CD}^{(n)} [b, b_1, \dots, b_k; c, c_{k+1}, \dots, c_n; x_1, \dots, x_n]$$

$$= \frac{1}{2\pi i} \frac{\Gamma(c)}{\Gamma(b)} \int_0^\infty \int_{-\infty}^{(0+)} e^{-(t-u)} t^{b-1} (1-x_1/u)^{-b_1} \dots (1-x_k/u)^{-b_k}$$

$$\cdot {}_0F_1 [-; c_{k+1}; tx_{k+1}] \dots {}_0F_1 [-; c_n; tx_n] dt du,$$

where $\text{Re}(b) > 0$.

$$(3.7) \quad {}_{(4)}^{(k)}\phi_{CD}^{(n)} [a, b, b_1, \dots, b_k; c; x_1, \dots, x_n]$$

$$= \frac{1}{2\pi i} \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \int_0^\infty \int_0^\infty \int_{-\infty}^{(0+)} e^{-(s+t-u)} e^{st(x_{k+1} + \dots + x_n)} s^{a-1} t^{b-1}$$

$$. u^{-c} (1-x_1 s/u)^{-b_1} \dots (1-x_k s/u)^{-b_k} ds dt du,$$

where $\text{Re}(a) > 0$, $\text{Re}(b) > 0$.

$$(3.8) \quad {}^{(k)}\phi_{CD}^{(n)} [a, b, b_1, \dots, b_k; c; x_1, \dots, x_n]$$

$$= \frac{1}{\Gamma(a) \Gamma(b) \Gamma(b_1) \dots \Gamma(b_k)} \int_0^\infty \dots (k+2) \dots \int_0^\infty e^{-s [1-t(x_{k+1} + \dots + x_{n-1}) + t_1 + \dots + t_k]} \\ \cdot s^{a-1} t^{b-1} t_1^{b_1-1} \dots t_k^{b_k-1} {}_0F_1 [-; c, x_1 s t_1 + \dots + x_k s t_k] \\ ds dt dt_1 \dots dt_k,$$

$\text{Re}(a) > 0$, $\text{Re}(b) > 0$, and $\text{Re}(b_i) > 0, i=1, \dots, k$,

$$(3.9) \quad {}^{(k)}F_{CD}^{(n)} [a, b, b_1, \dots, b_k; c, c_{k+1}, \dots, c_n; x_1, \dots, x_n]$$

$$= \frac{1}{\Gamma(a) \Gamma(b) \Gamma(b_1) \dots \Gamma(b_k)} \int_0^\infty \dots (k+2) \dots \int_0^\infty e^{-(s+t+t_1+\dots+t_k)} \\ \cdot s^{a-1} t^{b-1} t_1^{b_1-1} \dots t_k^{b_k-1} {}_0F_1 [-; c; s t_1 x_1 + \dots + s t_k x_k] \\ {}_0F_1 [-; c_{k+1}; s t x_{k+1}] {}_0F_1 [-; c_n; s t x_n] ds dt dt_1 \dots dt_k, \\ \text{Re}(a) > 0, \text{Re}(b) > 0, \text{Re}(b_i) > 0, i=1, \dots, k.$$

$$(3.10) \quad {}^{(k)}\phi_{CD}^{(n)} [a, b; c, c_{k+1}, \dots, c_n; x_1, \dots, x_n]$$

$$= \frac{1}{\Gamma(a) \Gamma(b)} \int_0^\infty \int_0^\infty s^{a-1} t^{b-1} e^{-(s+t)} {}_0F_1 [-; c; s x_1 + \dots + s x_k]$$

$$\cdot {}_0F_1[-; c_{k+1}; st_{k+1} x_{k+1}] \dots {}_0F_1[-; c_n; st_n x_n] ds dt,$$

$$\operatorname{Re}(a) \geq 0, \operatorname{Re}(b) \geq 0.$$

$$(3.11) \quad {}_{(2)}^{(k)}\phi_{CD}^{(n)}[a, b_1, \dots, b_k; c, c_{k+1}, \dots, c_n; x_1, \dots, x_k]$$

$$= \frac{1}{\Gamma(a) \Gamma(b_1) \dots \Gamma(b_k)} \int_0^\infty \dots (k+1) \dots \int_0^\infty e^{-(s+t_1+\dots+t_k)} s^{a-1} t_1^{b_1-1} \dots t_k^{b_k-1}$$

$${}_0F_1[-; c; x_1 t_1 s + \dots + x_k t_k s] {}_0F_1[-; c_{k+1}; x_{k+1} s] \dots {}_0F_1[-; c_n; x_n s] ds dt_1 \dots dt_k,$$

$$\operatorname{Re}(a) > 0, \operatorname{Re}(b_i) > 0, i=1, \dots, k.$$

$$(3.12) \quad {}_{(3)}^{(k)}\phi_{CD}^{(n)}[b, b_1, \dots, b_k; c, c_{k+1}, \dots, c_n; x_1, \dots, x_n]$$

$$= \frac{1}{\Gamma(b) \Gamma(b_1) \dots \Gamma(b_k)} \int_0^\infty \dots (k+1) \dots \int_0^\infty e^{-(t+t_1+\dots+t_k)} t^{b-1} t_1^{b_1-1} \dots t_k^{b_k-1}$$

$${}_0F_1[-; c; x_1 t_1 + \dots + x_k t_k] {}_0F_1[-; c_{k+1}; x_{k+1} t] \dots {}_0F_1[-; c_n; x_n t] dt dt_1 \dots dt_k,$$

$$\operatorname{Re}(b) > 0, \operatorname{Re}(b_i) > 0, i=1 \dots k.$$

4. RECURRENCE RELATIONS

Using the result due to Exton [3, p. 115 (3.5)]

$$(4.1) \quad {}_0F_1[-; c-1; x] - {}_0F_1[-; c; x] - \frac{x}{c(c-1)} {}_0F_1[-; c+1; x] = 0$$

we derive the following recurrence relations from (3.3), (3.4) and (3.5), respectively :

$$\begin{aligned}
 (4.2) \quad & {}^{(k)}F_{CD}^{(n)} [a, b, b_1, \dots, b_k; c, c_{k+1}, \dots, c_n; x_1, \dots, x_n] \\
 &= {}^{(k)}F_{CD}^{(n)} [a, b, b_1, \dots, b_k; c, c_{k+1}, \dots, c_{k+j-1}, c_{k+j}-1, c_{k+j+1}, \dots, \\
 &\quad c_n; x_1, \dots, x_n] \\
 &- \frac{x_{k+j}}{c_{k+j}(c_{k+j}-1)} {}^{(k)}F_{CD}^{(n)} [a, b, b_1, \dots, b_k; c, c_{k+1}, \dots, c_{k+j-1}, \\
 &\quad c_{k+j}+1, c_{k+j+1}, \dots, c_n; x_1, \dots, x_n]
 \end{aligned}$$

where $j=1, \dots, n-k$.

$$\begin{aligned}
 (4.3) \quad & {}^{(k)}\phi_{(1)CD}^{(n)} [a, b; c, c_{k+1}, \dots, c_n; x_1, \dots, x_n] \\
 &= {}^{(k)}\phi_{(1)CD}^{(n)} [a, b; c, c_{k+1}, \dots, c_{k+j-1}, c_{k+j}-1, c_{k+j+1}, \dots, c_n; \\
 &\quad x_1, \dots, x_n] \\
 &- \frac{x_{k+j}}{c_{k+j}(c_{k+j}-1)} {}^{(k)}\phi_{(1)CD}^{(n)} [a, b; c, c_{k+1}, \dots, c_{k+j-1}, c_{k+j}+1, \\
 &\quad c_{k+j+1}, \dots, c_n; x_1, \dots, x_n]
 \end{aligned}$$

where $j=1, \dots, n-k$.

$$\begin{aligned}
 (4.4) \quad & {}^{(k)}\phi_{(2)CD}^{(n)} [a, b_1, \dots, b_k; c, c_{k+1}, \dots, c_n; x_1, \dots, x_n] \\
 &= {}^{(k)}\phi_{(2)CD}^{(n)} [a, b_1, \dots, b_k; c, c_{k+1}, \dots, c_{k+j-1}, c_{k+j}-1, \\
 &\quad c_{k+j+1}, \dots, c_n; x_1, \dots, x_n]
 \end{aligned}$$

$$= \frac{x_{k+j}}{c_{k+j}(c_{k+j}-1)} {}^{(k)}\phi_{CD}^{(n)} [a, b_1, \dots, b_k; c, c_{k+1}, \dots, c_{k+j-1}, c_{k+j}+1, \\ c_{k+j+1}, \dots, c_n; x_1, \dots, x_n],$$

where $j=1, \dots, n-k$.

$$(4.5) \quad {}^{(k)}\phi_{CD}^{(n)} [b, b_1, \dots, b_k; c, c_{k+1}, \dots, c_n; x_1, \dots, x_n]$$

$$= {}^{(k)}\phi_{CD}^{(n)} [b, b_1, \dots, b_k; c; c_{k+1}, \dots, c_{k+j-1}, c_{k+j}-1,$$

$$c_{k+j+1}, \dots, c_n; x_1, \dots, x_n]$$

$$= \frac{x_{k+j}}{c_{k+j}(c_{k+j}-1)} {}^{(k)}\phi_{CD}^{(n)} [b, b_1, \dots, b_k; c, c_{k+1}, \dots, c_{k+j-1}, c_{k+j}+1,$$

$$c_{k+j+1}, \dots, c_n; x_1, \dots, x_n],$$

$j=1, \dots, n-k$.

5. GENERATING RELATIONS

From the definitions of Section 2, we obtain the following generating relations :

$$(5.1) \quad (1-t)^{-a} {}^{(k)}F_{CD}^{(n)} [a, b, b_1, \dots, b_k; c, c_{k+1}, \dots, c_n;$$

$$\frac{x_1}{1-t}, \dots, \frac{x_n}{1-t}]$$

$$= \sum_{r=0}^{\infty} \frac{(a)_r t^r}{r!} {}^{(k)}F_{CD}^{(n)} [a+r, b, b_1, \dots, b_k; c, c_{k+1}, \dots, c_n; x_1, \dots, x_n]$$

$$(5.2) \quad (1-t)^{-b} {}^{(k)}F_{CD}^{(n)} [a, b, b_1, \dots, b_k; c, c_{k+1}, \dots, c_n; x_1, \dots, x_k,$$

$$\frac{x_{k+1}}{1-t}, \dots, \frac{x_n}{1-t}]$$

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$$= \sum_{r=0}^{\infty} \frac{(b)_r t^r}{r!} {}^{(k)}F_{CD}^{(n)} [a, b+r, b_1, \dots, b_k; c, c_{k+1}, \dots, c_n; x_1, \dots, x_n],$$

$$(5.3) \quad (1-t)^{-b_i} {}^{(k)}F_{CD}^{(n)} [a, b, b_1, \dots, b_k; c, c_{k+1}, \dots, c_n; \\ x_1, \dots, x_{i-1}, \frac{x_i}{1-t}, x_{i+1}, \dots, x_k, \dots, x_n]$$

$$= \sum_{r=0}^{\infty} \frac{(b_i)_r t^r}{r!} {}^{(k)}F_{CD}^{(n)} [a, b, b_1, \dots, b_{i-1}, b_i+r, b_{i+1}, \dots, b_k; c, \\ c_{k+1}, \dots, c_n; x_1, \dots, x_n], \quad i=1, \dots, k,$$

$$(5.4) \quad (1-t)^{-a} {}^{(k)}\phi_{(1)CD}^{(n)} [a, b; c, c_{k+1}, \dots, c_n; \frac{x_1}{1-t}, \dots, \frac{x_n}{1-t}]$$

$$= \sum_{r=0}^{\infty} \frac{(a)_r t^r}{r!} {}^{(k)}\phi_{(1)CD}^{(n)} [a+r, b; c, c_{k+1}, \dots, c_n; x_1, \dots, x_n],$$

$$(5.5) \quad (1-t)^{-a} {}^{(k)}\phi_{(2)CD}^{(n)} [a, b_1, \dots, b_k; c, c_{k+1}, \dots, c_n;$$

$$\frac{x_1}{1-t}, \dots, \frac{x_n}{1-t}]$$

$$= \sum_{r=0}^{\infty} \frac{(a)_r t^r}{r!} {}^{(k)}\phi_{(2)CD}^{(n)} [a+r, b_1, \dots, b_k; c, c_{k+1}, \dots, c_n; x_1, \dots, x_n],$$

$$(5.6) \quad (1-t)^{-b} {}^{(k)}\phi_{(3)CD}^{(n)} [b, b_1, \dots, b_k; c, c_{k+1}, \dots, c_n; x_1, \dots, x_k,$$

$$\frac{x_{k+1}}{1-t}, \dots, \frac{x_n}{1-t}]$$

$$= \sum_{r=0}^{\infty} \frac{(b)_r t^r}{r!} {}^{(k)}\phi_{(3)CD}^{(n)} [b+r, b_1, \dots, b_k; c, c_{k+1}, \dots, c_n; x_1, \dots, x_n],$$

$$(5.7) \quad (1-t)^{-a} \phi_{(4)CD}^{(k)(n)} \left[a, b, b_1, \dots, b_k; c; \frac{x_1}{1-t}, \dots, \frac{x_n}{1-t} \right]$$

$$= \sum_{r=0}^{\infty} \frac{(a)_r t^r}{r!} \phi_{(4)CD}^{(k)(n)} [a+r, b, b_1, \dots, b_k; c; x_1, \dots, x_n].$$

$$(5.8) \quad (1-t)^{-b} \phi_{(1)CD}^{(k)(n)} \left[a, b; c, c_{k+1}, \dots, c_n; x_1, \dots, x_k, \frac{x_{k+1}}{1-t}, \dots, \frac{x_n}{1-t} \right]$$

$$= \sum_{r=0}^{\infty} \frac{(b)_r t^r}{r!} \phi_{(1)CD}^{(k)(n)} [a, b+r; c, c_{k+1}, \dots, c_n; x_1, \dots, x_n],$$

$$(5.9) \quad (1-t)^{-b} \phi_{(4)CD}^{(k)(n)} [a, b, b_1, \dots, b_k; c; x_1, \dots, x_k, \frac{x_{k+1}}{1-t}, \dots, \frac{x_n}{1-t}]$$

$$= \sum_{r=0}^{\infty} \frac{(b)_r t^r}{r!} \phi_{(4)CD}^{(k)(n)} [a, b+r, b_1, \dots, b_k; c; x_1, \dots, x_n],$$

$$(5.10) \quad (1-t)^{-b_i} \phi_{(2)CD}^{(k)(n)} [a, b_1, \dots, b_k; c, c_{k+1}, \dots, c_n; x_1, \dots, x_{i-1}, \frac{x_i}{1-t}, x_{i+1}, \dots, x_k, \dots, x_n]$$

$$= \sum_{r=0}^{\infty} \frac{(b_i)_r t^r}{r!} \phi_{(2)CD}^{(k)(n)} [a, b_1, \dots, b_{i-1}, b_i+r, b_{i+1}, \dots, b_k; c, c_{k+1}, \dots, c_n; x_1, \dots, x_n],$$

$$i=1, \dots, k.$$

$$(5.11) \quad (1-t)^{-b_i} \phi_{(3) CD}^{(k) (n)} [b, b_1, \dots, b_k; c, c_{k+1}, \dots, c_n; x_1, \dots, x_{i-1}, \\ \frac{x_i}{t}, x_{i+1}, \dots, x_k, \dots, x_n]$$

$$= \sum_{r=0}^{\infty} \frac{(b_i)_r t^r}{r!} \phi_{(3) CD}^{(k) (n)} [b, b_1, \dots, b_{i-1}, b_i + r, b_{i+1}, \dots, b_k; c, \\ c_{k+1}, \dots, c_n; x_1, \dots, x_n], \\ i=1, \dots, k.$$

$$(5.12) \quad (1-t)^{-b_i} \phi_{(4) CD}^{(k) (n)} [a, b, b_1, \dots, b_k; c, x_1, \dots, x_{i-1}, \\ \frac{x_i}{1-t}, x_{i+1}, \dots, x_k, \dots, x_n] \\ = \sum_{r=0}^{\infty} \frac{(b_i)_r t^r}{r!} [a, b, b_1, \dots, b_{i-1}, b_i + r, b_{i+1}, \dots, b_k; c; \\ x_1, \dots, x_n], \\ i=1, \dots, k.$$

In a forthcoming paper, we shall discuss fractional derivatives and Eulerian integral representations of the above functions.

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